MAPLE VERSUS A VARIABLE MASS SPRING EQUATION

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An undamped spring system is a well known topic in differential equations. However, the assumption that support of the mass instantly "turns off" at t=0 is flawed. In fact, when the spring is released from a non-equilibrium position there is a ramping up of the effect of gravity (via some function) during the short period of time during the release. Modeling the system's behavior during these short moments leads us to consider the following problem. What is the behavior of an undamped spring system for which the mass is increased linearly, and how can it be explained mathematically? We study two differential equations for this system with the use of Maple. The physical system we model is a bucket on a spring. Water flows into the bucket at a constant rate. We assume initial position and velocity are both zero and use the following parameters:

b = mass of the bucket, with spring mass correction
a = mass flow rate of water into the bucket
k = spring constant
g = acceleration due to gravity
wv = water velocity as it flows into the bucket

The most straightforward and easy to handle model is the standard differential equation for a spring, but with the additional mass due to the water flow treated as a forcing function. This leads us to the differential equation:

$$b\left(\frac{d^2}{dt^2}\mathrm{x}(t)\right) + k\,\mathrm{x}(t) = g\,a\,t$$

Solving this by hand is a straightforward application of second order, linear, constant coefficient differential equation theory. The DEtools package in Maple can also be used.

```
> ode0 := b*diff(x(t),t$2) + k*x(t)=g*a*t;
```

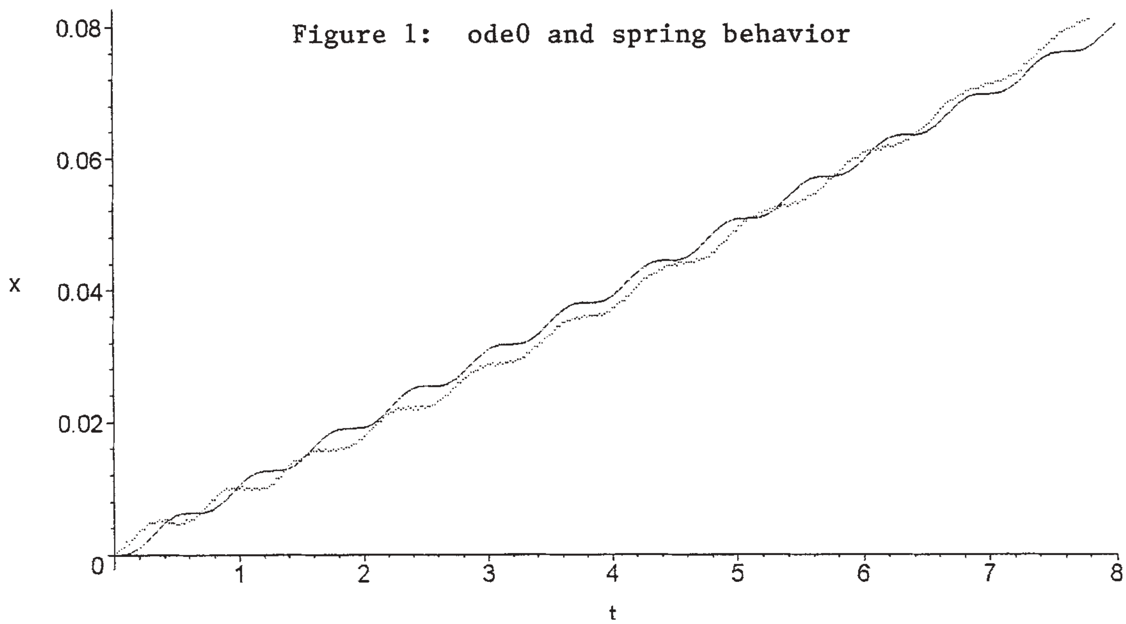
$$ode0 := b\left(\frac{d^2}{dt^2}x(t)\right) + k\,x(t) = g\,a\,t$$

```
> simplify(dsolve( {ode0, x(0)=0,D(x)(0)=0}, x(t)));
```

$$x(t) = \frac{\left(-\sin\left(\frac{\sqrt{k}\,t}{\sqrt{b}}\right)\sqrt{b} + \sqrt{k}\,t\right) g\,a}{k^{(3/2)}}$$

Unfortunately, checking this against data raises some questions.
The most distinctive departures of reality from the solution of ode0 are:
1) The interval of negative velocity near t=0, and
2) Over time the "plateaus" of 0 velocity do not occur and the curve appears to gradually smooth out.
We plot both curves on the same axes.

```
> DataPlot := PLOT(DataPoints): Sol0 := plot( rhs(dsolve( {ode0,
  x(0)=0,D(x)(0)=0}, x(t))),t=0..(8),labels=[t,x]):
  display( [DataPlot,Sol0]);
```



The real problem is that treating the increasing mass as a forcing function is a considerable oversimplification.
Variable mass systems are studied in Physics and techniques for modeling them are well known.

The correct relationship for a variable mass system is:

F = m x " - V m',

where V is the velocity of mass added relative to the velocity of the system.
The differential equation is then ode1:

$$(b+a\,t)\left(\frac{d^2}{dt^2}\mathrm{x}(t)\right)+a\left(\frac{d}{dt}\mathrm{x}(t)\right)+k\,\mathrm{x}(t)=g\,a\,t+a\,wv$$

Unfortunately, unlike the first model, this equation is not easily solvable without assistance.
Maple, however, not only solves the equation, but also yields unexpected insights.

```
> dsolve( ode1, x(t));
```

$$\mathrm{x}(t)=\mathrm{BesselJ}\left(0,\,2\sqrt{k}\sqrt{\frac{b+a\,t}{a^2}}\right)\_C2+\mathrm{BesselY}\left(0,\,2\sqrt{k}\sqrt{\frac{b+a\,t}{a^2}}\right)\_C1+\frac{a\,((g\,t+wv)\,k-g\,a)}{k^2}$$

Now that Maple has revealed the solution, it is possible to reproduce it "by hand" via the substitution

$$u=\frac{2\sqrt{k\,(b+a\,t)}}{a}$$

that transforms our equation into Bessel's equation of the 0-th order,

$$u^2\left(\frac{d^2}{du^2}\mathrm{x}(u)\right)+u\left(\frac{d}{du}\mathrm{x}(u)\right)+u^2\,\mathrm{x}(u)=0$$

Recall that the solution of ode1 is a linear combination of these two functions added to a line. Thus as t increases the oscillations of the Bessel functions produce flucuations around that line. The simpler model of ode0 has the same behavior but the oscillations are produced by sine and cosine rather than Bessel's functions.
Compare:

```
> dsolve( ode0, x(t));
```

$$\mathrm{x}(t)=\sin\left(\frac{\sqrt{k}\,t}{\sqrt{b}}\right)\_C2+\cos\left(\frac{\sqrt{k}\,t}{\sqrt{b}}\right)\_C1+\frac{g\,a\,t}{k}$$

```
> dsolve( ode1, x(t));
```

$$x(t) = \mathrm{BesselJ}\left(0, 2\sqrt{k}\sqrt{\frac{b+a\,t}{a^2}}\right)\_C2 + \mathrm{BesselY}\left(0, 2\sqrt{k}\sqrt{\frac{b+a\,t}{a^2}}\right)\_C1 + \frac{((g\,t+wv)\,k-g\,a)\,a}{k^2}$$

The slope of the line is the same in both cases. The most significant difference is the functions that govern the fluctuations around that line. The smoothing out of the plateaus in the data is a result of decreasing amplitude in those oscillating functions. A constant amplitude for sine and cosine is why they fail to describe the smoothing out. Bessel functions, however, do exhibit this behavior.

It's reasonable to ask how the quasi-period of the Bessel functions changes over time and how quickly the functions approach 0. These questions are easily answered by the following known properties of the Bessel functions. As x goes to infinity BesselJ(0,x) approaches

$$\sqrt{\frac{2}{\pi\,x}}\sin\left(x+\frac{\pi}{4}\right)$$

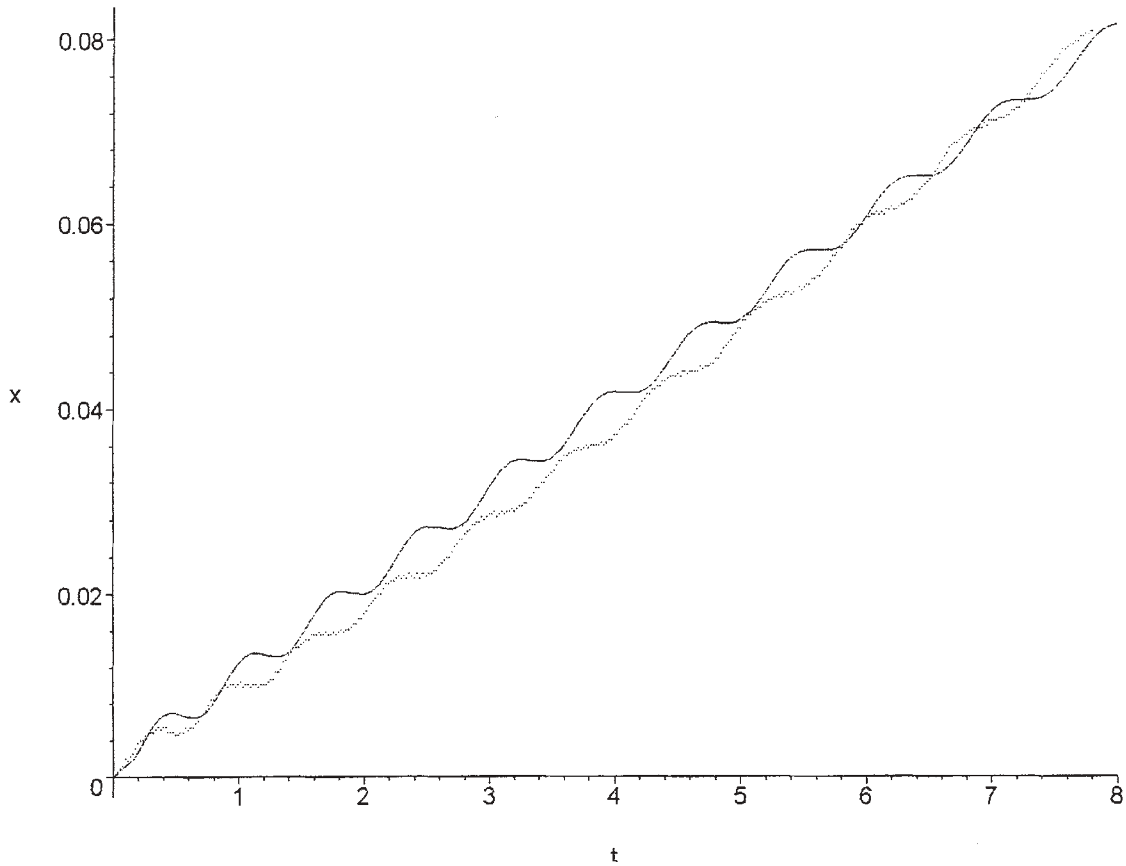
while BesselY(0,x) approaches

$$-\sqrt{\frac{2}{\pi\,x}}\cos\left(x+\frac{\pi}{4}\right)$$

Plotting the solution of ode1 with the data allows us to compare the two.

```
> Sol1 := plot( rhs(dsolve( {ode1, x(0)=0,D(x)(0)=0},
  x(t))),t=0..(8),labels=[t,x]):display([DataPlot,Sol1]);
```

Figure 2: odel and spring behavior



Because the functions do oscillate, the accumulation of experimental error makes the expectation of nearly matching curves somewhat unrealistic. However, elements of similarity that do support the model are:

1) Overall shape of the curves.
2) Slope of the line the curves oscillate around.
3) Long term decreasing amplitude of the oscillations.
4) Frequency of the oscillations.

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