USING MATHEMATICA® TO VISUALIZE PARTIAL DIFFERENTIAL EQUATIONS

Dr. Timothy Miller
Missouri Western State College
Department of Computer Science, Mathematics, and Physics
4525 Downs Drive
St. Joseph, MO 64507
milltim@mwsc.edu

Introduction

Mathematica® can be used to help students to visualize some of the important concepts in an introductory course in partial differential equations. While teaching such a course, I developed several demonstrations to illustrate the convergence of Fourier series, vibrating strings and membranes, heat flow, and the hanging chain problem.

The Vibrating String and the Method of D'Alembert

Consider a string with constant linear density that is stretched between two fixed points $x=0$ and $x=L$ on the *x*-axis. Let $u=u(x,t)$ be the transverse displacement of the string at *x* ($0<x<L$) at time *t*. It can be shown that $u=u(x,t)$ must satisfy the one-dimensional wave equation with boundary and initial conditions:

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2},\ \text{where } 0<x<L \text{ and } t>0,$$
$$u(0,t)=0 \text{ and } u(L,t)=0 \text{ for } t>0, \text{ and}$$
$$u(x,0)=f(x) \text{ and } \frac{\partial u}{\partial t}(x,0)=g(x), \text{ for } 0<x<L.$$

Here $c^2=\frac{\tau}{\rho}$ (τ is the tension in the string and ρ is the linear density of the string) and the *f* and *g* are given functions that describe the initial position and initial velocity of the string.

There are two ways we will solve this problem: (1) the standard separation of variables and express the solution as a Fourier sine series and (2) d'Alembert's solution that expresses the solution in terms of traveling waves:

$$u(x,t)=\frac{1}{2}\left[f^*(x-ct)+f^*(x+ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}g^*(s)\,ds,$$

where f^* and g^* are the odd periodic extensions of *f* and *g*.

Take $L=1$, $c=1$, and $g(x)=0$ for $0 \le x \le L=1$. It is easier to see the traveling waves if the function f is 0 for most of the unit interval. The function f will be the piecewise linear function given by

$$f(x)=\begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{4} \text{ or } \frac{1}{2} \le x \le 1 \\ 16x-4 & \text{if } \frac{1}{4} \le x \le \frac{5}{16} \\ -8x+\frac{7}{2} & \text{if } \frac{5}{16} \le x \le \frac{3}{8} \\ 4x-1 & \text{if } \frac{3}{8} \le x \le \frac{7}{16} \\ -12x+6 & \text{if } \frac{7}{16} \le x \le \frac{1}{2} \end{cases}$$

In Mathematica®, this piecewise-defined function can be entered as follows. The graph is given in Figure 1.

```
f[x_] := If[1/4 ≤ x ≤ 1/2, 1, 0] * If[1/4 ≤ x ≤ 3/8, Min[16x - 4, -8x + 7/2], Min[4x - 1, -12x + 6]]
```

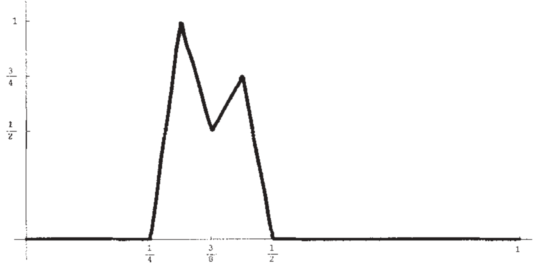

Figure 1 – The graph of f, the initial position of the string.

We now calculate the coefficients of the Fourier sine series for f.

$$\begin{aligned} a_n &= 2\int_0^1 f(x)\sin n\pi x\,dx \\ &= 2\left(\int_{\frac{1}{4}}^{\frac{5}{16}}(16x-4)\sin n\pi x\,dx + \int_{\frac{5}{16}}^{\frac{3}{8}}\left(-8x+\tfrac{7}{2}\right)\sin n\pi x\,dx \right. \\ &\qquad \left. + \int_{\frac{3}{8}}^{\frac{7}{16}}(4x-1)\sin n\pi x\,dx + \int_{\frac{7}{16}}^{\frac{1}{2}}(-12x+6)\sin n\pi x\,dx\right) \end{aligned}$$

After evaluating and simplifying, we obtain

$$a_n = \frac{-8}{\pi^2 n^2}\left(4\sin\tfrac{n\pi}{4} - 6\sin\tfrac{5n\pi}{16} + 3\sin\tfrac{3n\pi}{8} - 4\sin\tfrac{7n\pi}{16} + 3\sin\tfrac{n\pi}{2}\right)$$

The nth partial sum of the Fourier sine series is $\sum_{k=1}^{n} a_k \sin k\pi x$. The graphs of the partial sums of the Fourier sine series superimposed on the graph of f is given in Figure 2.

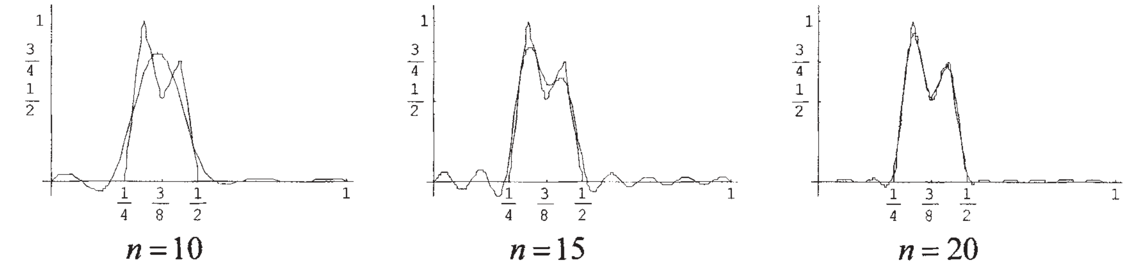

Figure 2 – Graphs of partial sums of the Fourier sine series of *f*.

We see that taking 20 terms of the series gives a good visual approximation to the initial position of the string. The solution of the problem will be $u(x,t)=\sum_{k=1}^{\infty} a_k \sin k\pi x \cos k\pi t$, use the first 20 terms for the approximate solution to graph. An animation can be shown using Mathematica® by using the command:

```
Do[Plot[Evaluate[Σ_{k=1}^{20} a[k] Sin[k π x] Cos[k π t], {x, 0, 1}, PlotRange → {-1, 1}]], {t, 0, 2, .05}]
```

Figure 3 shows the displacement at $t = 0.0,\ 0.2,\ 0.4,\ 0.6,\ 0.8$, and 1.0.

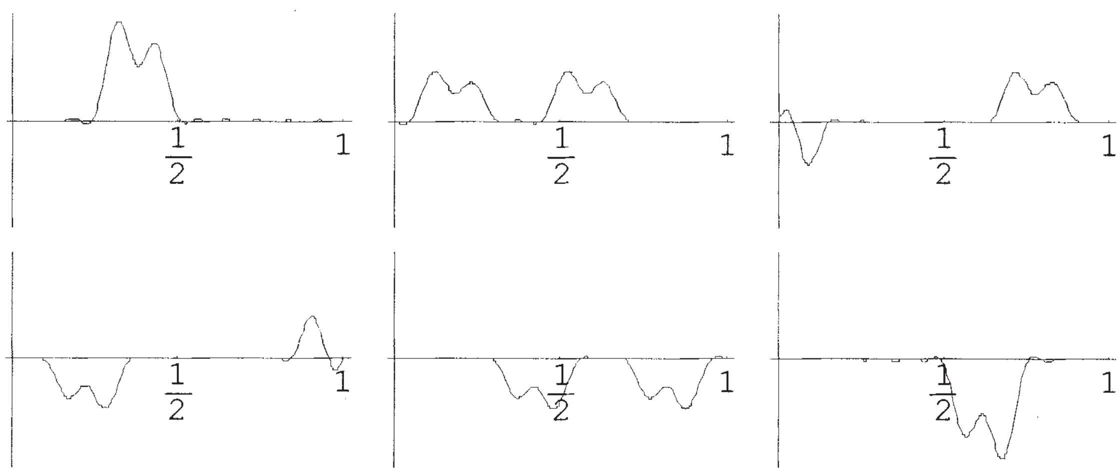

Figure 3 – The displacement of the string.

By looking at the d'Alembert method we get a new insight into the behavior of the vibration of the string. For our problem, the d'Alembert method gives the solution in the form $u(x,t)=\frac{1}{2}\left[f^*(x-t)+f^*(x+t)\right]$, where f^* denotes the odd periodic extension of *f*. The first term, $f^*(x-t)$, represents this extension of *f* moving to the right and the second term, $f^*(x+t)$, represents this extension of *f* moving to the left. Figure 4 shows the graph of $u(x,t)=\frac{1}{2}\left[f^*(x-t)+f^*(x+t)\right]$ for $t = 0.0,\ 0.2,\ 0.4,\ 0.6,\ 0.8$, and 1.0.

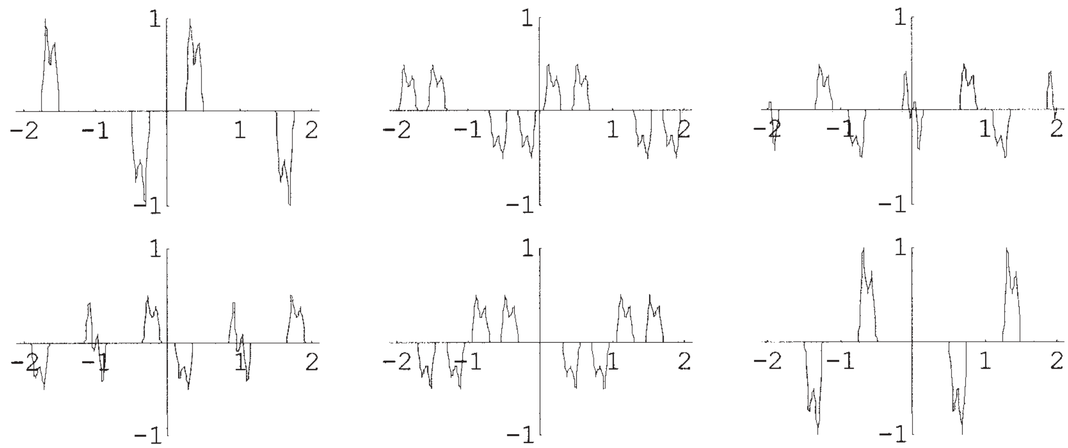

Figure 4 – The graph of $u(x,t)=\frac{1}{2}\left[f^{*}(x-t)+f^{*}(x+t)\right]$.

The Hanging Chain Problem

A chain with uniform density is hanging from a support. The x-axis is vertical and $x=0$ at the bottom of the chain; $x=L$ at the top. The u-axis if horizontal and the transverse movements of the chain are in the xu-plane. The differential equation is given by

$$\frac{\partial^2 u}{\partial t^2}=g\left(x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\right),$$

$$u(L,t)=0,\ \text{for } t>0,\ \text{and}$$

$$u(x,0)=f(x)\ \text{and}\ \frac{\partial u}{\partial t}(x,0)=h(x),\ \text{for } 0<x<L,$$

where g is the gravitational acceleration and f and h are given functions. Take $g=L=1$, $h(x)=0$, $f(x)=\begin{cases}.01 & \text{if } 0\le x\le 0.5\\ 0.02(1-x) & \text{if } 0.5\le x\le 1\end{cases}$. By the separation of variables method, we see that the solution will be a Bessel series. The solution of the problem is given by the series:

$$u(x,t)=\sum_{j=1}^{\infty}\frac{0.04\left(2J_2(\alpha_j)-J_2(\alpha_j\sqrt{.5})\right)}{\left(\alpha_j J_1(\alpha_j)\right)^2}J_0\left(\alpha_j\sqrt{x}\right)\cos\left(\frac{\alpha_j}{2}t\right),$$

where J_0, J_1, and J_2 are the Bessel functions of order 0, 1, and 2, respectively, and α_j is the j^{th} positive zero of the Bessel function of order 0. Use Mathematica® to obtain these Bessel coefficients and the solution (for $t=0$).

```
<< NumericalMath`BesselZeros`
α = BesselJZeros[0, 50];
A = Table[.04/(α[[j]] BesselJ[1, α[[j]]])^2 (2 BesselJ[2, α[[j]]] - BesselJ[2, α[[j]] √.5]), {j, 1, 50}];
u[x_, n_] := Sum[A[[j]] BesselJ[0, α[[j]] √x], {j, 1, n}]
```

Using Mathematica® as before, it appears that using 15 terms will give a good visual approximation to the initial position of the chain.

The following Mathematica® commands will give an animation of the movement of the hanging chain:

```
u[x_, t_] := Sum[(.04/(α[[j]] BesselJ[1, α[[j]]])^2 (2 BesselJ[2, α[[j]]] - BesselJ[2, α[[j]] √.5]) *
        BesselJ[0, α[[j]] √x] Cos[α[[j]]/2 t]), {j, 1, 15}]

Do[ParametricPlot[{u[x, t], x}, {x, 0, 1}, PlotRange→{{-.015, .015}, {0, 1}}, Ticks→
        {{-.01, .01}, {0, .5, 1}}, PlotStyle→{Hue[1]}], {t, 0, 6, .1}]
```

Figure 5 shows the position of the chain at $t = 0.0,\ 0.5,\ 1.0,\ 1.5,\ 2.0$, and 2.5.

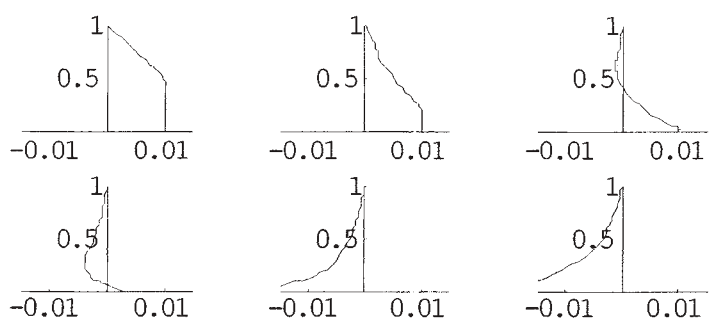

Figure 5 – The position of the chain.

In conclusion, the study of the partial differential equations of mathematical physics offers a rich environment for the use of the Mathematica® to show the connection between the mathematics and the physical model.

References:

(1) N. Asmar, *Partial Differential Equations and Boundary Value Problems*, Prentice-Hall, 2000

(2) S. Wagon, *Mathematica® in Action*, 2nd edition, Springer-Verlag, 1999

(3) S. Wolfram, *The Mathematica® Book*, 4th edition, Wolfram Media/Cambridge University Press, 1999