

RISKY BUSINESS: CONNECTING
MATHEMATICS AND BUSINESS CONCEPTS

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There are a number of imponderables in this world, such as “Why do clocks run clockwise?” and “Why is 100% alcohol called 200 proof?” Another imponderable is “How can we convince business and economics majors that mathematics is important to their futures?” This job is harder because many members of business and economics departments are not convinced about the value of calculus in their programs.

We know from our experiences with introductory economics courses that economists like to use lines to illustrate concepts visually, even when the concepts are not modeled well by linear functions. Thus we may maintain students interest by using an occasional visual argument.

Our goal is to show that the average cost per unit is minimized when the average cost equals the marginal cost. Suppose that the total cost function for a product is

$$C = 100 + 4x + \frac{x^2}{4}$$

Then the average cost function is

$$\bar{C} = \frac{100}{x} + 4 + \frac{x}{4}$$

and the derivative of this function is

$$\bar{C}' = -\frac{100}{x^2} + \frac{1}{4}$$

Setting this derivative equal to 0 and solving gives

$$\frac{100}{x^2} = \frac{1}{4} \quad \text{or} \quad x = \pm 20$$

Of course, we can show that the minimum occurs at (20, 280) and the minimum average cost is 280 units when 20 units are produced.

Now consider the marginal cost function for this product,

$$\overline{MC} = 4 + \frac{x}{2}$$

The average cost \bar{C} equals the marginal cost \overline{MC} where

$$\frac{100}{x} + 4 + \frac{x}{4} = 4 + \frac{x}{2}, \quad \text{or where} \quad 400 + 16x + x^2 = 16x + 2x^2$$

This gives

$$400 = x^2 \quad \text{or} \quad x = \pm 20$$

and we see that the values of x that satisfy $\bar{C}' = 0$ also satisfy $\bar{C} = \overline{MC}$.

We can easily show that this is true for a general quadratic cost function, and we can show that it is true for any polynomial cost function. For the cost function

$$C = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_ix^i + \dots + a_nx^n$$

$$\bar{C} = \frac{a_0}{x} + a_1 + a_2x + a_3x^2 + \dots + a_ix^{i-1} + \dots + a_nx^{n-1}$$

and

$$\bar{C}' = -\frac{a_0}{x^2} + a_2 + 2a_3x + \dots + a_i(i-1)x^{i-2} + \dots + a_n(n-1)x^{n-2}$$

So if average cost has a minimum, it occurs at a solution to the equation

$$0 = -\frac{a_0}{x^2} + a_2 + 2a_3x + \dots + a_i(i-1)x^{i-2} + \dots + a_n(n-1)x^{n-2}$$

or, for $x \neq 0$, to

$$a_2x^2 + 2a_3x^3 + \dots + (i-1)a_ix^i + \dots + (n-1)a_nx^n - a_0 = 0$$

For this cost function, the marginal cost function is

$$\overline{MC} = a_1 + 2a_2x + 3a_3x^2 + \dots + ia_ix^{i-1} + \dots + na_nx^{n-1}.$$

Then $\bar{C} = \overline{MC}$ where

$$\frac{a_0}{x} + a_1 + a_2x + a_3x^2 + \dots + a_ix^{i-1} + \dots + a_nx^{n-1} =$$

$$a_1 + 2a_2x + 3a_3x^2 + \dots + ia_ix^{i-1} + \dots + na_nx^{n-1}$$

This gives

$$a_2x + 2a_3x^2 + \dots + (i-1)a_ix^{i-1} + \dots + (n-1)a_nx^{n-1} - \frac{a_0}{x} = 0$$

or, for $x \neq 0$,

$$a_2x^2 + 2a_3x^3 + \dots + (i-1)a_ix^i + \dots + (n-1)a_nx^n - a_0 = 0$$

This equation is equivalent to the equation whose solution gives

$$C' = 0,$$

so if a minimum average cost occurs, it occurs where

$$\bar{C} = \overline{MC}.$$

But business and economics majors will be bored with this discussion, and it is incomplete. So let's try an argument that is more general, and that even economists would appreciate.

Suppose that we have a total cost function for which the average cost function is continuous and has a minimum in the first quadrant, so that the minimum average cost and the number of units giving it are positive.

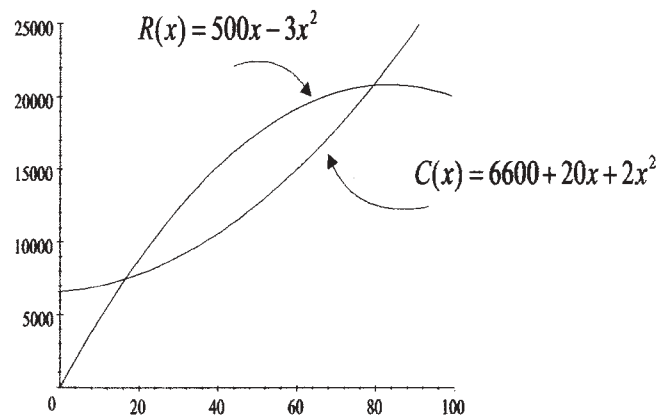
The average cost when x_1 units are produced and sold is

$$\frac{C(x_1)}{x_1}.$$

But this is the slope of the line from the origin to $(x_1, C(x_1))$.

The slope of the line from the origin to $(x_2, C(x_2))$ is less than the slope to $(x_1, C(x_1))$, so the average cost is less. The point on the curve where the slope is smallest is the point that gives the minimum average cost. If a line hits the curve twice, there is a point on the curve where the slope is smaller, and the slope is not a minimum. The line with the smallest slope that hits the graph is tangent to the curve at the point where the minimum occurs. Thus this line is also the tangent line from the origin to the point and has the same slope. That is, the minimum average cost occurs at a point where the average cost equals the marginal cost.

We can use a similar visual argument to discuss profit maximization. Consider the cost and revenue functions for the production and sale of a product.



We can see two breakeven points, and we know that there is some point between the breakeven points where the difference $R(x) - C(x)$ is a maximum, and it is where the distance between the two curves is the largest. If we draw tangent lines to the curves at different values of x (the number of units produced and sold), we see several possibilities.

1. For some values of x , the slope of the tangent line to the revenue curve is greater than the slope of the tangent line to the cost curve. In this case, increasing the value of x will increase the revenue faster than the cost, so the revenue is growing faster than the cost and the profit is still increasing. That is, profit will increase if more units are produced and sold, and profit has not yet been maximized.
2. For some values of x , the slope of the tangent line to the revenue curve is smaller than the slope of the tangent line to the cost curve. In this case, increasing the value of x will increase the cost faster than the revenue, so the revenue is growing more slowly than the cost profit will decrease as x increases. This means that the profit will increase if fewer units are produced and sold, and profit is not maximized.
3. Thus we seek the value of x where the tangent lines are parallel; that is, where the slopes of the tangent lines are equal. But this is where $R'(x) = C'(x)$, or where marginal revenue equals marginal cost.

In general, the x -value where $\overline{MC} = \overline{MR}$ between the breakeven points gives the maximum profit for this function.

The Equation Solver in the math menu of the TI-83 calculator can be used to solve future value problems and present value problems, using the formulas for future value and present value. But even more useful is the TVM (Time Value of Money) Solver in the FINANCE menu. This feature allows us to enter values for some variables of the equations mentioned above and to quickly solve for a remaining variable without entering the equations.

INVESTMENTS

A pair of twins enter the workforce at age 21. The first twin invests \$2400 per year for the first 7 years, and stops investing, but lets her money grow until she is 65 years old. The second twin does not invest for the first 7 years, but then invests \$2400 per year for the next 37 years, until he is 65 years old. If both investments average 10% interest per year, compounded annually:

1. How much money will each twin have at age 65?
2. When will the second twin's investment equal that of the first twin?
3. If the investments average 11%, when will the second twin's investment equal the first twin's?

The following formulas are used to solve this problem.

If \$ R is deposited at the end of each period for n periods in an annuity that pays interest at a rate of i per period, the future value of the annuity will be

$$S = R \cdot \left[\frac{(1+i)^n - 1}{i} \right].$$

If \$ P is invested for t periods at a rate of i per period, compounded each period, the future value S at the end of the n th period is $S = P(1+i)^n$

The solution follows.

1. How much money will each twin have at age 65?

Twin 1

$$2400 \left[\frac{(1+.10)^7 - 1}{.10} \right] = 22,769.21$$

$$FV = 22,769.21(1+.10)^x$$

At $x = 37$, FV is \$774,243.06

The first twin has \$22,769.21 at the end of the seven years, after investing \$16,800.

At age 65, $x = 37$, giving a value of \$774,243.06.

Twin 2

$$FV = 2400 \left[\frac{(1+.10)^x - 1}{.10} \right]$$

At $x = 37$, FV = \$792,094.77 Twin 2 invests 88,800 over 37 years, giving a value of \$792,094.77 at age 65.

Thus twin 2 has a little more money at age 65, but he has invested much more.

2. When will the second twin's investment equal that of the first twin?

By graphing both of these investments on the same graphing window, we can see, roughly, where the curves intersect. By using trace, we can see that the curves intersect near 30 years. We can check the values of the investments by using the TVM menu. At 31 years, the investments have grown to \$437,040.02 and \$436,664.22, respectively.

Using SOLVER with the two equations shows that the investment of Twin 2 catches that of Twin 1 after 31.165585846 years, where each investment has \$443,992