GEOMETRY AND MATRIX GROUPS IN LINEAR ALGEBRA

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We present two activities using the software Derive to integrate geometric and group theoretic notions into a linear algebra course. Linear algebra is a natural place to introduce transformational geometry, symmetry, and the concept of a group via matrix multiplication and matrix actions on vector spaces. The value of making these connections of linear algebra to geometry and group theory is twofold. These activities provide additional geometric content to the mathematics major and ease the transition to more abstract courses in algebra. Moreover activities such as those presented in this paper can help students realize that mathematics is not a collection of disjoint topics but rather a field with many interrelated topics which can often best be understood through multiple mathematical viewpoints.

Geometry is often given a minimal emphasis, if any, in many undergraduate mathematics programs. By placing some geometric content in linear algebra, students not only have the opportunity to visit geometry in an undergraduate mathematics course beyond calculus but also may deepen their understanding of the algebraic content. Moreover, an algebraic viewpoint of transformational geometry and symmetry would nicely supplement other approaches of these topics provided in modern geometry courses, which is of particular importance to pre-service high school teachers.

The algebraic concept underlying these geometric concepts is that of a group. Linear algebra is often taught as a computationally oriented course or as a course designed to help students make the transition from computational courses to more abstract, theoretical courses in the mathematics major. In particular, linear algebra helps students to prepare for a course in abstract algebra in which group theory is likely to be the primary topic. Even though the primary algebraic object studied in linear algebra, a vector space, is indeed a group, making the connection between the abstract concepts of vector space and group can be difficult for students in a first course in abstract algebra. Providing a brief, computationally concrete, geometric introduction to the concept of a group using the familiar objects of matrices and vectors can help students ease the transition into the abstract algebra class as well as solidify connections between algebra and geometry.

The Symmetry Group of a Regular Hexagon

This activity explores the symmetry group of a regular hexagon using matrices. The activity introduces students to the notion of using a matrix to define a function on (a subset of) $\mathbf{R}^2$. In particular, students use the following rotation and reflection matrices:

$$\mathbf{Rot}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \mathbf{Refangle}_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix},$$

where $\mathbf{Rot}_\theta$ is the counterclockwise rotation of θ about the origin, and $\mathbf{Refangle}_\theta$ is the reflection in the line through the origin rotated through a counterclockwise angle of θ from the x–axis. Students create and plot a regular hexagon and then use basic geometry (or vector geometry) to verify that their hexagons are indeed regular. After determining the symmetries of the hexagon, the students are guided to painlessly use Derive to create the group table for the symmetry group. The students then use the table to verify that this set of symmetries is indeed a group. For a further exercise, students can be guided to create a presentation for this dihedral group. We now present the activity step by step and provide several comments about Derive implementation.

1. *We must first set the input mode in Derive to* `WORD`. *To do this, Select the* Declare *menu, the* Input Settings *sub-menu, and then click to select* Word Mode *and* OK. *Now define the Derive functions for the rotation and reflection matrices. To do this enter the following lines:*

   $\texttt{ROT}(\theta) := [\texttt{COS}(\theta), -\texttt{SIN}(\theta); \texttt{SIN}(\theta), \texttt{COS}(\theta)]$
   $\texttt{Refangle}(\theta) := [\texttt{COS}(2\theta), \texttt{SIN}(2\theta); \texttt{SIN}(2\theta), -\texttt{COS}(2\theta)]$

2. *Plot a regular hexagon inscribed in the unit circle. You can do this by starting with one vertex at the point $P = (1, 0)$. Using the appropriate rotation matrix (using* $\texttt{ROT}(\theta)$*), find the other five vertices of the hexagon. Once you have the vertices of the hexagon, you may connect them with line segments to form the hexagon.*

   To accomplish this, we can use the following lines of code.

   $\texttt{P} := [1, 0]$
   $\texttt{R1} := \texttt{ROT}\left(\frac{\pi}{3}\right)$
   $\texttt{VECTOR}(\texttt{R1}^{\texttt{n}}\texttt{P}`, \texttt{R1}^{(\texttt{n}+1)}\texttt{P}`, \texttt{n}, 0, 6)$

   We then obtain the table of points

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -1 & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & 0 \end{bmatrix}$$

Now go to a plot window and make sure the display options for points are set to *Large* and *Connected*. The resulting figure will be the desired hexagon.

3. *Either using results from your text and from basic geometry or by direct computation, show that the hexagon that you have created is indeed regular; that is, the lengths of the sides of the hexagon are equal, and the angles of the hexagon are equal.*

   Students can perform these computations by hand or with Derive using the distance formula and the cosine formula for the dot product. Alternatively, students can use the fact the rotation used to plot the hexagon is an isometry of the hexagon and hence preserves distance and angles.

4. *Use* ROT(θ) *and* Refangle(θ) *to determine all of the symmetries (expressed as matrices) of a regular hexagon. How many are there?*

5. *Now that you have developed a list of all of the symmetries of the regular hexagon, the next task is to express all of them in terms of products and powers of just TWO symmetries. State the two symmetries that you will use, and then express each of the other symmetries in terms of these two.*

   The students should find matrices for the five rotations, six reflections, and the identity transformation. An example of a pair of transformations which is generate all twelve transformations is given by $r = \texttt{ROT}\left(\frac{\pi}{3}\right)$ and $s = \texttt{Refangle(0)}$. A presentation for this symmetry group is

$$\langle r, s \mid rs = sr^5 \rangle.$$

6. *Let D denote the set of these symmetries. Your next step is to create a multiplication table for these symmetries (expressed as matrices). Use the following type of Derive command to create the table:*

$$\texttt{VECTOR}([\texttt{m} * \texttt{a}_1, \texttt{m} * \texttt{a}_2, \ldots, \texttt{m} * \texttt{a}_n], \texttt{m}, [\texttt{a}_1, \texttt{a}_2, \ldots, \texttt{a}_n]),$$

2. *Note that each vertex on this cube lies on the sphere of radius* $\sqrt{3}$ *centered at the origin. Moreover, the symmetry group of any sphere centered at the origin is* O(3)*, with orientation preserving subgroup* SO(3)*. These are the groups of orthogonal and special orthogonal* 3×3 *matrices, respectively. Thus, the orientation preserving symmetry group of the cube can be represented as a subgroup of* SO(3).

3. *Find and plot the three 4-fold axes of symmetry of the cube, and find a matrix representation of a* $1/4-$*turn rotation about each axis in* SO(3).

4. *Find and the plot four 3-fold axes of symmetry, and find a matrix representation of a* $1/3-$*turn rotation about each axis is* SO(3).

   This step is more complicated than the previous step and requires several important techniques from linear algebra. To do this, first find an orthonormal basis $B = \{\mathbf{n}, \mathbf{q}_1, \mathbf{q}_2\}$ for $\mathbf{R}^3$, where $\mathbf{n}$ is a unit vector in the direction of the axis of rotation and $\mathbf{q}_1$ and $\mathbf{q}_2$ form an orthonormal basis of a plane preserved by the rotation. In particular, find (and plot) an equilateral triangle on the cube that is preserved by a $1/3-$turn rotation about the given axis. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two vertices of this triangle, and verify that set $B = \{\mathbf{n}, \mathbf{v}_1, \mathbf{v}_2\}$ is basis for $\mathbf{R}^3$. Now apply the Gramm-Schmitt orthonormalization process to B to obtain an orthonormal basis $B' = \{\mathbf{n}, \mathbf{q}_1, \mathbf{q}_2\}$. Interchange $\mathbf{q}_1$ and $\mathbf{q}_2$, if necessary, so that the determinant of the matrix with columns $\mathbf{n}$, $\mathbf{q}_1$, and $\mathbf{q}_2$ has a determinant of 1.

   The matrix of the $1/3-$turn rotation with respect to the basis B' is

   $$M'_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ 0 & \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix},$$

   and the change of basis matrix from B' to the standard basis is $P = [\mathbf{n}\,\mathbf{q_1}\,\mathbf{q_2}]$. The desired matrix of the $1/3-$turn is then $M = PM_{B'}P^{-1}$.

5. *There are six axes of order two joining midpoints of opposite sides of the cube. Find and plot these axes. Then determine a matrix representation for each of these half-turn rotations.*

6. *Construct the multiplication table for this collection of transformations and verify that it is indeed a group.*

7. *Show that this group is isomorphic to the symmetric group* S_4*. That is, create a one-to-one correspondence between these groups that preserves the group structure.*

*where $a_1, a_2, \ldots, a_n$ are the matrices representing the symmetries of the regular hexagon. The ij*th *entry in your table will be the matrix product $a_i a_2$. You should note that each product in this table is indeed an element of the set D. We can conclude that D is* closed *under matrix multiplication. Note that even though we are using matrices to represent these symmetries, each symmetry is really a function of the hexagon. Thus, the operation of matrix multiplication corresponds to the composition of the symmetries.*

7. *Use the fact that matrix multiplication is associative to show that matrix multiplication in D is associative.*

8. *Find an element I of D such that $AI = A = IA$ for each A in D. The element I is called the* identity element *of D under the operation of matrix multiplication.*

9. *Show that each element of D has an inverse in D. That is, show that for each matrix A in D, there is a matrix A^{-1} in D such that $AA^{-1} = A^{-1}A = I$. This step and the proceeding three steps prove that D is a* group *under the operation of matrix multiplication.*

10. *Matrix multiplication is not commutative in D. Give an example of two elements of D that do not commute. Illustrate this graphically.*

    The best way to study this table is to print it out and tape the pieces of paper together. It is also recommended to label the rows and columns of this table with the appropriate matrices. From the table, the students should then be able to observe closure, identify the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as the identity, and identify multiplicative inverses. Students can also verify that this group is not commutative from the table.

The Symmetry Group of a Cube

This activity is similar in spirit to the symmetry group of the hexagon activity, yet there are some features that make this activity significantly more difficult. Understanding this problem geometrically requires the student to be able to visualize the three-dimensional geometric behavior. A computer algebra system can easy help the visualization process. The step which requires students to find matrices representing $1/3-$rotations about the axes of three-fold symmetry is best approached by finding each matrix with respect to a convenient orthonormal basis for $\mathbf{R}^3$ using the Gramm-Schmitt process and then by applying the change of basis theorem to express each matrix in terms of the standard basis for $\mathbf{R}^3$.

1. *Plot the cube in $\mathbf{R}^3$ with vertices at* $(\pm 1, \pm 1, \pm 1)$, $(\mp 1, \pm 1, \pm 1)$, $(\pm 1, \mp 1, \pm 1)$, and $(\pm 1, \pm 1, \mp 1)$.