INTERACTIVE MODULES FOR DIFFERENTIAL AND INTEGRAL CALCULUS OF FUNCTIONS OF MANY VARIABLES

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Introduction

Many students find multi-variable calculus very challenging. Not only does this course introduce many difficult concepts (e.g., curvature, Jacobian, curl, etc.), but it also requires that students have a good grasp of necessary prerequisite mathematical techniques (which, unfortunately, is far from certain).

Furthermore, the problem is exacerbated by many students inability to visualize objects in three (or even two) dimensional space. Many problems either require a geometric insight as a part of the solution (e.g., to properly set up iterated integral limits), or can benefit from a geometric interpretation in addition to the analytic solution.

Over the last ten years or so, the authors have created Mathcad modules that not only allow students to focus on the true problem at hand and improve their problem-solving skills, but also afford them some geometric insight about the multi-variable calculus concept or technique being discussed.

In this article, we will discuss some examples of such modules, which take advantage of Mathcad's Electronic Book interface [1].

Triple Integrals

In the course of this assignment, students are asked to convert given triple integrals to another coordinate system (rectangular to cylindrical and spherical, as well as cylindrical to spherical). Specifically, one of the problems (see Figure 1), involves rewriting the given cylindrical coordinate integral in spherical coordinates.

Problem

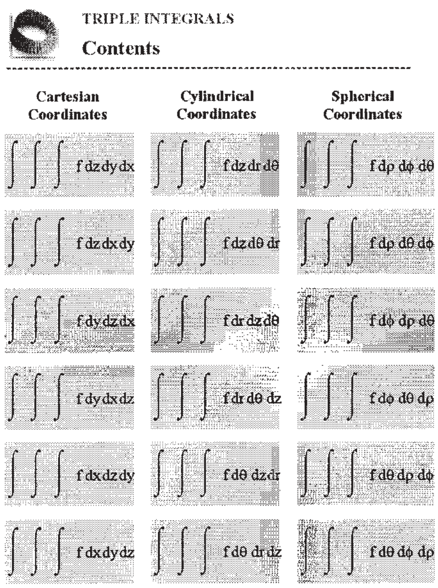
Given the triple integral $\int_0^{\frac{\pi}{2}}\int_0^{4}\int_{-r}^{\frac{r}{\sqrt{3}}} r^3 \, dz \, dr \, d\theta$

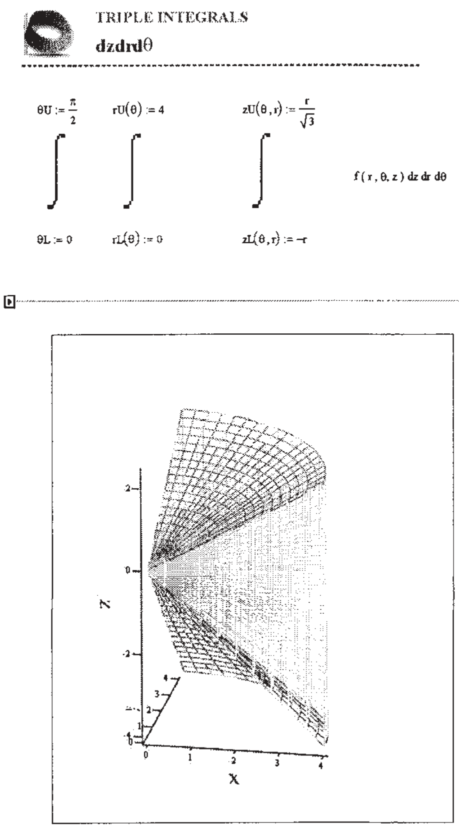
- Use the **Triple Integrals Handbook** to plot the region of integration in cylindrical coordinates. Paste the *entire handbook page* into your report.
- Evaluate the integral symbolically using Mathcad's <Ctrl>-. operator.
- By hand, convert the integral to **spherical coordinates**.
- Use the **Triple Integrals Handbook** to plot the region of integration in spherical coordinates. Make sure the region matches the original region. Paste the *entire handbook page* into your report.
- Evaluate the spherical integral symbolically using Mathcad's <Ctrl>-. operator. Make sure the answer agrees with the original integral value.

Figure 1.

A sample problem from the Triple Integrals lab assignment.

Figure 2.
(a) Triple Integrals Handbook table of contents (below)
(b) Region of integration set up in cylindrical coordinates (right)





While working on this assignment, students are given access to the "Triple Integrals Handbook", designed to visualize regions of integration (not the integrand) for a triple integral set up in one of the three coordinate systems.

Initially, a student would select the proper format of the cylindrical integral from the handbook's table of contents (Figure 2(a)), then proceed to specify the limits of the iterated integrals to obtain a visual display of the region of integration (Figure 2(b)).

Evaluating the integral symbolically in Mathcad yields

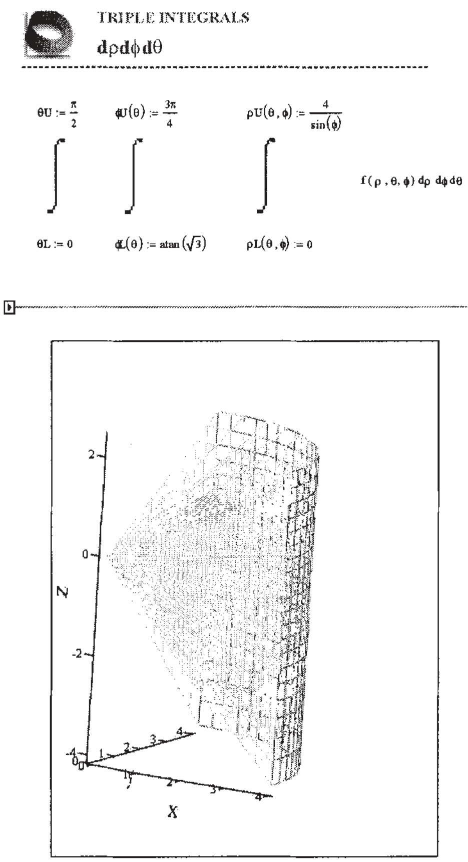
$$\int_0^{\frac{\pi}{2}} \int_0^4 \int_{-r}^{\frac{r}{\sqrt{3}}} r^3 \, dz \, dr \, d\theta \rightarrow \frac{512}{15} \cdot \pi \cdot 3^{\frac{1}{2}} + \frac{512}{5} \cdot \pi$$

The students would now proceed to work out the details of converting the limits and the integrands by hand. At this point in the course, the topic of triple integrals in different coordinate systems has just recently been introduced to the students. Consequently, this task is expected to be quite challenging to most students – in fact, many of them will fail to obtain a correct answer at their first attempt.

It is therefore pedagogically important that **constructive** guidance be provided to those students whose initial attempt is unsuccessful. This is why the lab activity asks the student to proceed to visualize the new (spherical coordinate) region using the Triple Integrals Handbook **before** the new integral is evaluated. Here are some advantages of this approach:

- Students can focus on one thing at a time: converting the region of integration can be done first, and independent of converting the integrand.
- Most importantly, a mistake made when converting the region of integration will result in a three-dimensional region that does not visually match the original region. Students are encouraged to inspect the discrepancies, and try to eliminate them by correcting the mistakes they have made. Some students may go through more than one "trial and error" iteration before arriving at the correct region. This is a much better type of feedback than just "right" or "wrong" which is provided by simply evaluating the integral.

Figure 3.
The region of integration of Figure 2(b) correctly converted to spherical coordinates.



Of course, once they confirm that the region appears to have been correctly converted to the new coordinate system, students turn their attention to the integrand. (They have to be particularly careful when converting between cylindrical and spherical systems to properly account for dV in both coordinate systems). Having completed this step correctly, they would obtain the following result from Mathcad symbolics:

$$\int_0^{\frac{\pi}{2}}\int_{\operatorname{atan}(\sqrt{3})}^{3\cdot\frac{\pi}{4}}\int_0^{\frac{4}{\sin(\phi)}} \rho^4\cdot\sin(\phi)^3\,d\rho\,d\phi\,d\theta \rightarrow \frac{512}{15}\cdot\pi\cdot 3^{\frac{1}{2}}+\frac{512}{5}\cdot\pi$$

Space Curves

Another use of the electronic book format is as interactive reference guide. As an example, a chapter from the *Curves and Surfaces* handbook is illustrated.

Chapter 4. Velocity, Acceleration and Curvature

- Position Vector
- Velocity and Acceleration Vectors
- T, N and B
- T, N (and B) in the plane
- Curvature
- Curvature in the Plane
- Circle of Curvature
- Circle of Curvature in the Plane

Figure 4.

Table of contents of a sample chapter from the Curves and Surfaces handbook.

Clicking on any of the section headings hyperlinks the student links to an interactive document, that not only reviews and defines the appropriate mathematical topic, but also has an interactive graphic helping the student to visualize the topic. For example, clicking on the T, N and B section, the student will find both the definitions and alternative methods of determining these vectors. In addition, there is a space curve defined and a point on the curve specified. Mathcad calculates the vectors at the point, and plots both the space curve and T, N and B, as shown in Figure 5.

The students are able to see how the vectors change as the point traverses the curve. Furthermore, they are able to look at different problems by changing the definition of the space curve. In this way, they can get visual feedback to assigned homework problems.

As a final example, we show a laboratory assignment from the chapter on space curves. The problem is to determine the tangent lines to a given curve at two distinct points. Students are then asked to show the lines intersect, and find the point of intersection. Using the handbook, students are able to quickly calculate and plot both tangent lines. They can then focus on determining the point of intersection of the lines. The document

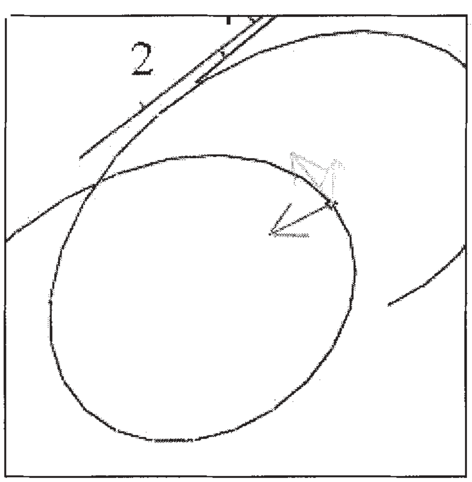
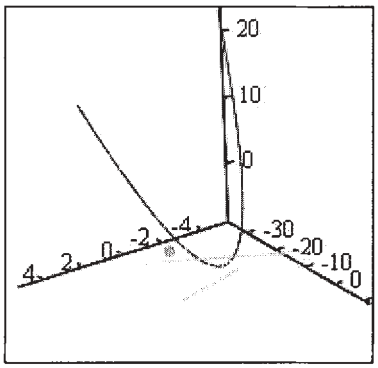


Figure 5.

The plot of a space curve, and the vectors T, N and B.

Calculate the tangent lines to the space curve $r(t) := \begin{pmatrix} t \\ 2t - t^2 \\ t^2 - 1 \end{pmatrix}$ at the points given by t=0 and t=2.

Find the point of intersection of the tangent lines. Change the highlighted equations below to illustrate your answer.



r,L1,L2,P

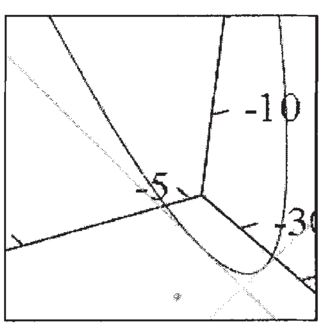
Figure 6.

The problem defined with incorrect formulae for tangent lines.

they are provided has an incorrect point plotted. By putting in their value for the point of intersection, they get instant visual feedback from the graph whether it is correct as illustrated in Figures 6 – 7. With the visual feedback, we find students are more determined, and more successful, with this type of problem than if traditional paper and pencil methods were used.

References

[1] G. Melrose and P. Bogacki, *Creating and Using Electronic Books with Mathcad,* Proceedings of the Thirteenth International Conference on Technology in Collegiate Mathematics, November 2000, Atlanta, Georgia, pp. 242-245

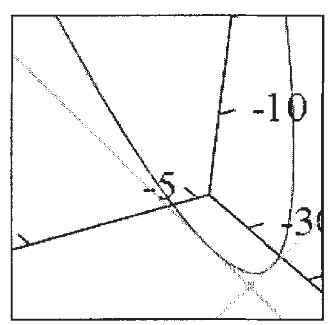


r,L1,L2,P

Figure 7.

(a - left) the tangent lines have been correctly determined,

(b - right) the point of intersection has now been correctly determined.



r,L1,L2,P