

Interactive Web-Based Calculus Projects at Hollins University: Area of U.S. States and Roller Coasters

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In the mathematics and statistics curriculums at Hollins University, there is much emphasis on the application of concepts to real life problems. For this purpose, we have recently written two interactive web-based calculus projects which use JAVA applets as well as downloadable Maple worksheets. These projects are original and exciting and have been well received by both students and colleagues.

Design of a Thrilling Roller Coaster

In this Calculus I project (www1.hollins.edu/depts/math/hammer/coaster/), students use differentiation concepts to mathematically model the path of roller coasters and to analyze the path of these coasters with regard to peak and valley points (local extreme points), points of steepest ascent and descent (inflection points), and “thrill.” Students first apply these ideas to interactive on-line images of real coasters and then extend these ideas to their own coaster designs which must satisfy the following coaster restrictions:

- The total horizontal length of the coaster must be less than 200 feet.
- The track starts 75 feet above ground and ends at ground level.
- The track cannot exceed a height of 75 feet or go below ground level.
- No ascent or descent can be steeper than 80 degrees with the horizontal.
- The roller coaster must start and end with a zero degree incline.
- The thrill of a drop is the product of the radian measure of the angle of steepest descent with the vertical distance of the drop.
- The thrill of the coaster is the sum of the thrills of the drops.
- The path of the coaster must be modeled by a differentiable function.

Students use downloadable Maple worksheets to design and mathematically build the straight stretch coasters and to determine the corresponding thrill. Cubic polynomial functions and trigonometric functions are used to model the paths of these single drop and several drop coasters. As a project assignment, students are asked to design a coaster which satisfies the given restrictions and that has the maximum thrill. This optimization problem is a challenging one but the interactive coaster windows and the downloadable Maple worksheets make it easy (and fun!) for students to experiment with different designs. This project was developed by Professor Trish Hammer and student Jessica King from Hollins University and Professor Steve Hammer from Virginia Western Community College with support from the Hollins faculty/student research fund.

Students follow a step by step procedure which begins with a basic introduction to coaster design. In this introductory section, students gain an overall understanding of the

methods being used. An interactive web window and Java applet allow students to simply point and click to mark peak and valley points. The window automatically displays the cubic polynomial which fits their marked points along with the slope at the steepest points. Students then use this information to calculate the thrill of each drop and the total thrill for the coaster.

In this first section, students do not actually determine mathematical models. They simply see the mathematical models in action. Figure 1 below shows the automatically generated piecewise defined cubic polynomials that fit (in height and slope) the marked peak and valley point of one portion of the Steel Dragon. Students use the displayed slopes and y coordinates to calculate thrill.



$$\text{angle1} = \arctan(1.254) = 0.8976 \text{ radians} = 51.4 \text{ degrees.}$$

$$\text{angle2} = \arctan(1.0675) = 0.8180 \text{ radians} = 46.9 \text{ degrees.}$$

$$\text{angle3} = \arctan(1.34) = 0.9297 \text{ radians} = 53.3 \text{ degrees.}$$

$$\text{angle4} = \arctan(0.8579) = 0.7091 \text{ radians} = 40.6 \text{ degrees.}$$

$$\text{Thrill} = (217-115)*(0.8976)+(189-121)*(0.9297) = 154.7748$$

Figure 1
(coaster image from www.coastergallery.com)

Next, students begin to mathematically model the path of the coasters. Data in the form of peak and valley points is collected from the live on-line images of real life coasters and then students import these data points into a downloadable Maple worksheet. The Maple worksheet is complete with the commands (plot, solve, differentiate, do loops) necessary for visualization and mathematical computation and analysis of the mathematical models. We choose to use actual worksheets rather than Maplets because students at Hollins are expected to learn to read, write and understand Maple code.

Four modeling methods are presented. Students begin with modeling just a single coaster hill and then progress to modeling a roller coaster with several hills. In each case, students are asked to use cubic polynomials and then trigonometric functions. For each method, students are given specific step by step instructions that lead them through the mathematical modeling process. At the end of each of the four sections, students are asked to model their own coaster path by first creating an interesting design and then by specifying peak and valley points on the path. Their goal is to design a coaster path which obeys the given restrictions (regarding height, slope, and differentiability) and which is also thrilling.

For example, to mathematically model several drops of the Steel Dragon coaster, students enter the collected x - and y -coordinates of the peak and valley points (marked in Figure 1) into a downloadable Maple worksheet. They then use Maple commands to fit a cubic polynomial (or a trigonometric function) to each consecutive pair of points by creating a system of four equations in four unknowns. Two equations result from the matching of y -coordinates and two equations result from the matching of slopes (first derivatives.) With Maple, students are able to quickly graph their piecewise mathematical model to see if it does in fact match the real coaster image. Students then use first and second derivatives to identify the points of steepest ascent and descent (points of inflection) and the corresponding slope at these points. The slopes are then converted to corresponding angles (with the horizontal) and students compute the thrill of each drop of the coaster, and finally the total thrill of the coaster. A student generated mathematical model for the Steel Dragon path marked in Figure 1 is shown below in Figure 2. The degree measures of the angles of steepest descent and ascent are also displayed. “SAFE” indicates that no ascent or descent is steeper than 80 degrees with the horizontal.

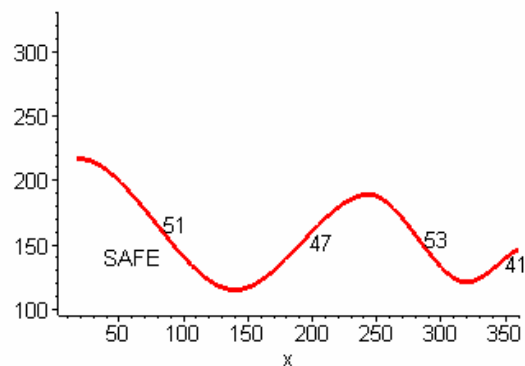


Figure 2

In the last section of the project, students must complete a project assignment. In this assignment, students are asked to create a coaster that satisfies all coaster restrictions and that is the most thrilling. This optimization project is a challenging one. How do they create the “most thrilling” coaster? Should they use many shorter hills or fewer taller hills? The solution to this optimization problem requires the use of non-peak/valley points and non-zero slope conditions, so students must think carefully about how to modify their approach and the Maple code. For this project assignment, students must submit mini-technical reports that contain a complete mathematical explanation of their optimal design along with the Maple worksheets.

A preliminary version of this project has been used for many years at Hollins University and at Virginia Western Community College. In this version, students were shown an example of a mathematical model (using Maple) for a roller coaster and then asked to model a roller coaster that was most thrilling. Students who were “good” at Maple found the project exciting and were able to experiment easily to find the best coaster design. Those students who were not as “good” at Maple struggled, found the project (and Maple syntax) frustrating and were just happy to produce one mathematical model. We believe

the addition of the interactive images that allow point and click experimentation along with easily modifiable Maple code will allow all students a quick and easy way to check out their “hunches.” They will then be able concentrate on mathematical concepts and not on Maple syntax.

Estimating the Area of US States

Hollins uses an entirely different project in the Calculus II course. This module involves the use of integration approximation techniques for the computation of the area of the state of Virginia (http://www1.hollins.edu/depts/math/Prep/area_of_virginia.htm). The module was developed by Hollins faculty as part of an on-line PREP workshop sponsored by the MAA and NSF in July 2002, and was recently published in [The Journal of Mathematics and Its Applications](#). The module was co-written by Julie Clark, Trish Hammer, and Caren Diefenderfer of Hollins University, and by Steve Hammer, of Virginia Western Community College.

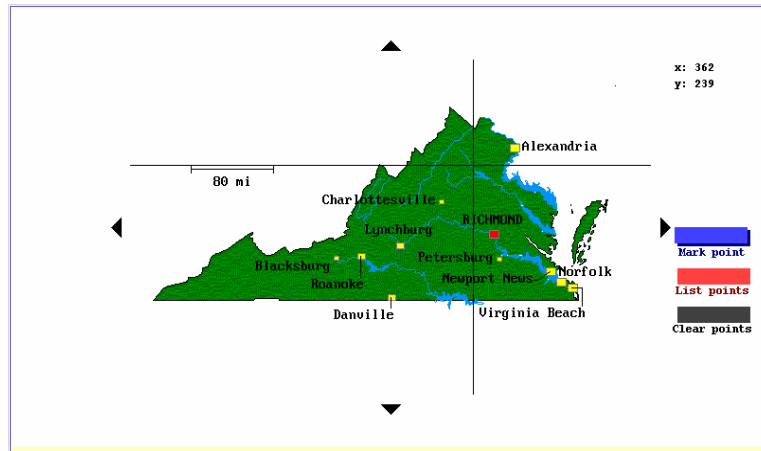
In this project, students use a "live" on-line map of Virginia to collect data points along the boundaries of the state, and then they import these data points into a downloadable Maple worksheet. This Maple worksheet is complete with the commands (plot, solve, integrate, do loops) necessary for visual and mathematical construction of the area approximations.

Four approximation methods are presented in this project; they progress from the simpler techniques (rectangular and trapezoidal) which are handled by geometric formulas to a more complex technique (Simpson's Rule) which involves integration of approximating quadratic functions. Instructors may choose to cover the combination of approximation methods that seem most appropriate for their course and students in any particular semester. For example, instructors might assign either the left or right rectangular approximation method, or both, or they might assign only the Simpson's Rule approximation method.

For each method, students are given specific step by step instructions for the area calculations based on the collection of 10 data points. Students are then asked to improve their approximations by modifying both their data collection and the corresponding Maple8 worksheet to include 30 or more data points.

For example, to use trapezoids to approximate the area, students are first directed to the following web page (<http://www1.hollins.edu/depts/math/Prep/VAMap2.htm>) in order to learn to use the interactive map of Virginia, and to collect 10 pairs of coordinates to use in their subsequent work. A screen shot of the live map is shown in Figure 3 below.

Figure 3



Using this Live Map

- Click anyplace on the map to move the cursor to that point.
- Notice the readout at the upper right gives the location of the cursor measured in pixels from the lower left corner of the map.
- You can fine tune the location of the cursor by clicking on the four arrows at the left, right, top, and bottom of the map. Each click moves the cursor one pixel in the direction indicated by the arrow.

(The map and applet used in the Area of Virginia project were obtained at <http://g-lea.tamu.edu/smap.htm>, and http://www.dean.usma.edu/math/people/wattenberg/Lite_Applets_JOMA/afghanistan_distances.html)

Students are then directed to open the provided MAPLE worksheet, and to store their collected coordinates in the worksheet. The students use MAPLE to plot their coordinates and they are encouraged to re-sample until their plot provides a reasonable outline of the state of Virginia. The worksheet then defines the 10 linear functions used to construct the trapezoid approximations. It also provides code that displays a plot of the imported pairs of coordinates, along with the ten approximating trapezoids. Subsequent commands compute the area of each trapezoid, and the sum of all the areas. The trapezoids drawn by MAPLE are actually too large – they include a rectangular region below the state as shown in Figure 4 below. So students are directed to subtract the appropriate rectangular area from their trapezoidal sum.

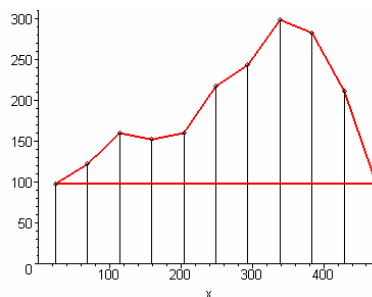


Figure 4

This MAPLE worksheet also includes commands for using integrals to compute the sum of the areas of the individual trapezoids. Of course such integration is not necessary in this case, but will be required when students move to using parabolas in the Simpson's rule section of the project, and so understanding these ideas now is an excellent precursor.

Finally, the students use the map and MAPLE worksheet to determine the number of pixels equivalent to 80 miles. They then determine the conversion factor from square pixels to square miles and arrive at a final estimate for the area of Virginia. Students are then asked to re-examine their entire work, summarize it and argue whether their approximation provides an over- or under-estimate of the actual area of Virginia. They are then asked to increase the accuracy of their approximation by using more (at least 30) trapezoids and by altering the MAPLE worksheet to adjust for the additional trapezoids.

Once students have worked through as many of the four approximation methods as the instructor desires, they are given a final project assignment in which they must approximate the area of one of the states California, Illinois, Maryland or North Carolina ("live" maps are provided). These states were chosen so that students must think carefully about how they should modify and extend the previous (Virginia) procedure. For example, the nice horizontal border in these states is the northern (rather than southern) border. Also, each of the four states has some "indentation" in its shape. We expect the end result of the project to be a well-written paper that includes a complete explanation and justification of the student's modified procedure as well as complete Maple worksheets.

Student solutions for this final project have been quite varied and creative. We highlight just a few of these solutions below. For the state of California, students begin by using the interactive map (Figure 5) to mark points along the boundary.



Figure 5

One student chose to sample two distinct sets of boundary points (red and blue) as shown in Figure 6 below. She then used Maple to calculate the area of the red trapezoids and the area of the blue trapezoids. She then subtracted the blue area from the red to arrive at a final approximation. Note that she chose to sample a great many points so as to increase her accuracy.

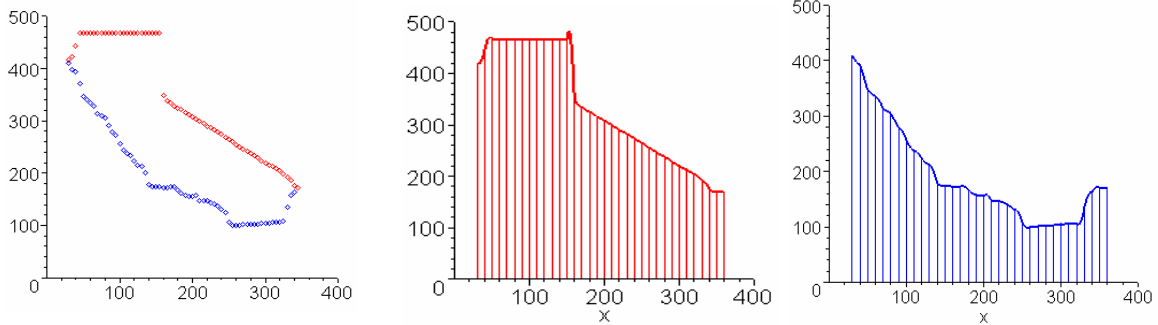


Figure 6

Another student chose a similar, but somewhat different approach. Rather than constructing trapezoids with horizontal bases along the x -axis, she turned her trapezoids “upside down” and used a line along the north edge of the state as the base of the horizontal trapezoids. (She must have enjoyed using the live-map to sample points because she sampled a very large number of them!) She then calculated blue trapezoid area minus red trapezoid area to approximate the area of California.

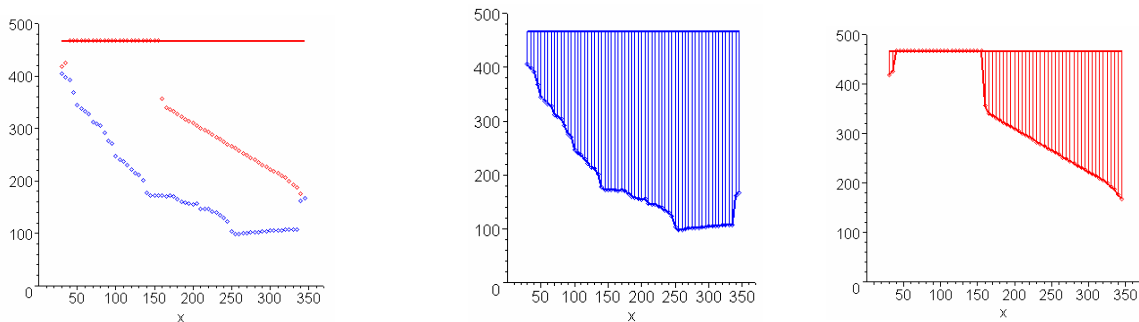


Figure 7

A third student took an entirely different approach to California (see Figure 8). She viewed the northern border of the state as a horizontal line and decided that the eastern edge could be considered a combination of a vertical and slanted line. Thus she sampled points only along the southwestern edges, used Maple to find the area of a large rectangle, and then subtracted from this rectangle’s area the area of a (blue) trapezoid and the area of the red trapezoids shown below. Her approach meant she did not need to use the live-map to sample nearly as many coordinate pairs.

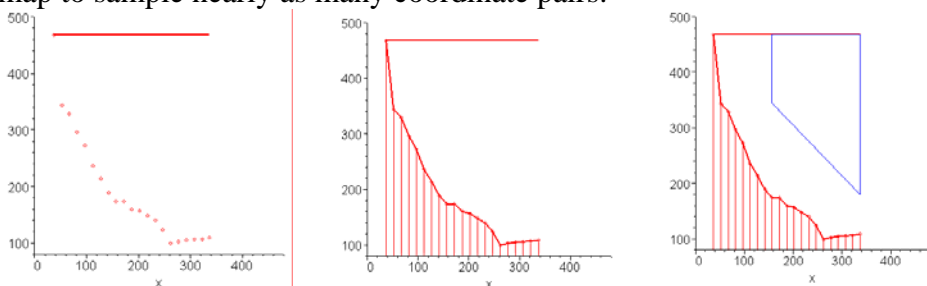


Figure 8

Figure 9 below shows how one student chose to estimate the area of North Carolina. She wanted to take advantage of both horizontal segments along the borders, so she used Simpson's Rule to estimate the area of the blue region and then subtract the area of the blue rectangular region. Then she found the area of the red rectangular region, used Simpson's rule to find the area of the red region below the state boundary, and subtracted this from the area of the red rectangle. Then she added the areas of the blue and red state regions.

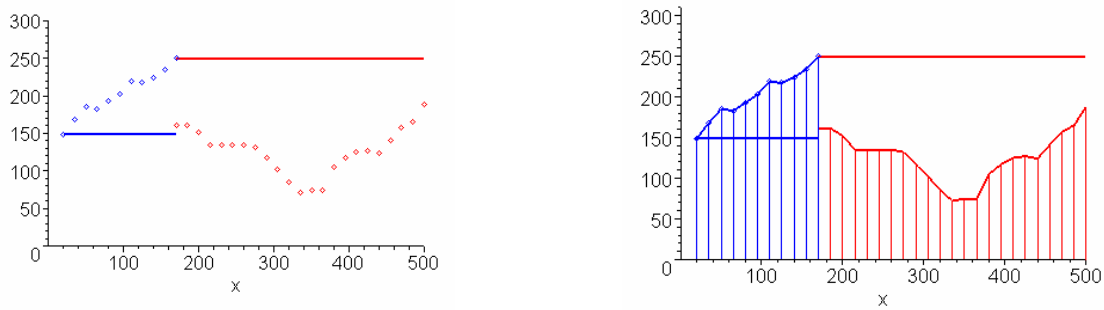


Figure 9

In summary, we were very pleased with the student work on these projects. Students enjoyed the challenges and seemed particularly proud of their creative and colorful methods of solution. We believe their success was due to the organization of the project, where they were led carefully through the process for Virginia and through the initial Maple code, and then asked to think creatively about how to apply their ideas to other irregularly shaped states. With the availability of the interactive state maps and easy to modify Maple code, students found it easy (and fun!) to experiment with many different possible solutions and to see (almost immediately) their techniques in action. They were able to focus on the method of solution and not on Maple syntax.

Complete solutions for this module are available from the Hollins Department of Mathematics and Statistics. Contact Phammer@hollins.edu for more details.