

Exploring Parabolic Mirrors with TI92 Plus/TI89

Vlajko L. Kocic

Department of Mathematics, Xavier University of Louisiana, 1 Drexel Dr., New Orleans, LA 70125, e-mail: vkocic@xula.edu

Reflecting telescope, flashlights, satellite dish antenna, optical illusions like “levitating penny” and “inverted light bulb”... What do they have in common? The answer is in the title of this paper: parabolic mirror. What is a parabolic mirror? It is a mirror whose reflecting surface is in the shape of a paraboloid (surface whose all cross sections are parabolas). Why parabolic mirror? Well, it has a property that the incident rays parallel to the axis of the parabola (for example, those that are coming from distant objects, like planets, stars, galaxies, or electromagnetic signals from satellites) after being reflected from the mirror, pass through the same point – the focus of a parabola.

This paper focuses on the reflection from a parabolic mirror and involves activities designed for students enrolled in at least Calculus I. Part III deals with solving differential equations and is suitable for some Calculus II students (where basic differential equations topics are covered) or for students taking Differential Equations course. Part I is suitable as a classroom activity, while part II is more appropriate as a homework assignment or project. Part III might be used as an introduction to Lagrange and Clairaut type differential equations (if they are covered in the course). TI92 Plus scripts for solving Lagrange and Clairaut types of differential equations are given in the Appendix.

In this paper we provide the TI92 Plus script for all activities. The script has been selected because it is easy to follow and to modify so students may explore.

This activity involves and helps to develop a number of different skills and techniques: implicit differentiation, symbolic manipulations including substitutions, graphing families of curves, graphing parametric equations, and solving differential equations.

First let us recall the Law of Reflection that states: *The angle of incidence is equal to the angle of reflection.* In other words, when the light hits the flat surface it reflects from the surface in such a way that the angle between the incident ray and normal to the surface is the same as the angle between the reflected ray and the normal to the surface. The incident ray, the reflected ray and the normal all three lie in the same plane (see Fig. 1). The law applies to flat surfaces as well as to curved surfaces.

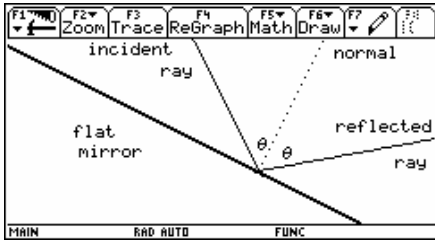


Figure 1 - Reflection of light

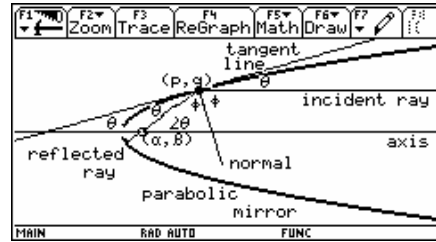


Figure 2 - Parabolic mirror

Now, consider the parabolic mirror (that is, in fact, a paraboloid whose cross sections are parabolas). The reflection of rays parallel to the horizontal axis of the parabola is illustrated in Fig. 2. Let m be a slope of the tangent line to the parabola at the point (p, q) and let n be the slope of the reflected ray. Then, clearly

$$m = \tan \theta \quad \text{and} \quad n = \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2m}{1 - m^2}.$$

The problem of studying reflection properties of parabolas can be roughly divided into the following three parts:

Part I. *A parabolic mirror was constructed so the cross section of the mirror has an equation $x = ay^2 + by + c$. A series of light waves moves parallel to the axis and are reflected off the surface of the parabola. Use TI-92+/TI-89 to graphically verify the reflection property of the parabola: For a series of parallel incident rays, all reflected waves pass through the same point, the focus of the parabola. Also, use TI-92+/TI-89 to analytically prove the reflection property of parabola.*

Part II. *Explore whether other reflecting surfaces have similar properties. For example, using TI-92+/TI-89 graphically explore if the spherical mirror whose cross section has the equation $x = -\sqrt{c^2 - y^2}$ possesses the similar reflection property.*

Part III. *The natural question arises: Is the parabola the only smooth curve with such property? Find all smooth curves with the property that all reflected rays intersect at a single point. Without loss of generality we may take that the point of intersection is the origin.*

TI92 Plus script for parabolic mirror in Part I can also be used (with minor modifications) to study a reflection from a curved mirror, described with general equation. Some commands in the script are included to ensure easy transition to more general equation and play no role in the case of parabola. The script in Part II points the parts that need to be modified when exploring the reflection from a spherical mirror. It is always useful to keep one copy of the Part I script and make changes into another copy (use CopyVar command to copy text variables).

```

:Reflection of the series of parallel l
ight waves from the surface whose cros
s section has an equation is f(x,y)=0
C:ClrHome:ClrGraph
C:setMode("Split Screen","Left-Right"
"Split 1 App","Text Editor","Split 2 Ap
p","Home")
C:setMode("Graph","Function")
C:DelVar a,b,c,x,y,p,q,m,n,r,f1,f2,f3,f4
:Define the function and use the implic
it differentiation to find the slope o
f the tangent line m at the point (p,q)
on the curve
C:Define f(x,y)=a*y^2+b*y+c-x
C:d(f(x,y(x))=0,x)
C:ans(1)d(y(x),x)=m and y(x)=q
C:solve(ans(1),m)
C:ans(1)|x=p
C:right(ans(1))=m
:Finding the slope of the reflected ray
and the equation of reflected ray
C:tExpand(tan(2*tan^4(m)))=n
C:solve(f(p,q)=0,p)
C:right(ans(1))=p
:f1 and f2 - equations of the mirror;
f3 - Incident ray; f4 - Reflected ray
C:Define f4(x)=q+n*(x-p)
C:Define f3(x)=q
C:expList(solve(f(x,y)=0,y),y)
C:Define f1(x)=ans(1)[1]
C:Define f2(x)=ans(2)[2]
:Selecting values for parameters of a m
irror (if necessary) and the family of
incident rays
C:1+a:0+b:-4+c:{-2,-1,0,1,2}=q
:Review functions prior to graphing
C:f1(x)
C:f2(x)
C:f3(x)
C:f4(x)
:Window and format settings
C:4*xmin:2*xmax:-3*ymin:3*ymax
C:setGraph("Coordinates","Off")
C:setGraph("Axes","Off")
:Graphing
C:graph f1(x):graph f2(x):graph f3(x):gr
aph f4(x):shade ymin,f1(x),xmin,xmax,2
1:shade f2(x),ymax,xmin,xmax,2,1
:The proof that all reflected rays pass
through the same point: the intersect
ion of any two reflected rays does not
depend on the selection of rays
C:DelVar a,b,c,q
C:Define f5(x)=f4(x)|q=r
C:expList(solve(f4(x)=f5(x)|q=r,x),x)
C:f4(ans(1)[1])

```

Figure 3 - Part I TI92 Plus script

- Use implicit differentiation to find $\frac{dy}{dx}$ of the implicit function $f(x,y)=0$. The slope of a tangent line at (p,q) is $m = \frac{dy}{dx}\bigg|_{(p,q)}$ and the slope of reflected ray is $n = \tan(2\arctan(m))$.
- If p is a solution $f(p,q) = 0$, so the reflected ray of the incoming ray $f_3(x) = q$ has an equation $f_4(x) = q + n(x - p)$.
- Mirror curves $f_1(x)$ and $f_2(x)$ are solutions of the equation $f(x,y) = 0$ with respect to y .
- Select values for parameters a, b, c , and q , appropriate graph and window settings. Prior to graphing, review functions $f_1(x), f_2(x), f_3(x)$, and $f_4(x)$. If necessary, make adjustments in the Home screen.
- Graph functions and shade the regions above and below the mirror curves.
- Delete variables, define another reflected ray $f_5(x)$ and show that the intersection of $f_4(x)$ and $f_5(x)$ is independent on reflected rays.

The following calculator screens illustrate the above script. When the script is executing it will appear in the split screen (Text and Home screens). Here it is presented full Home screen for better appearance.

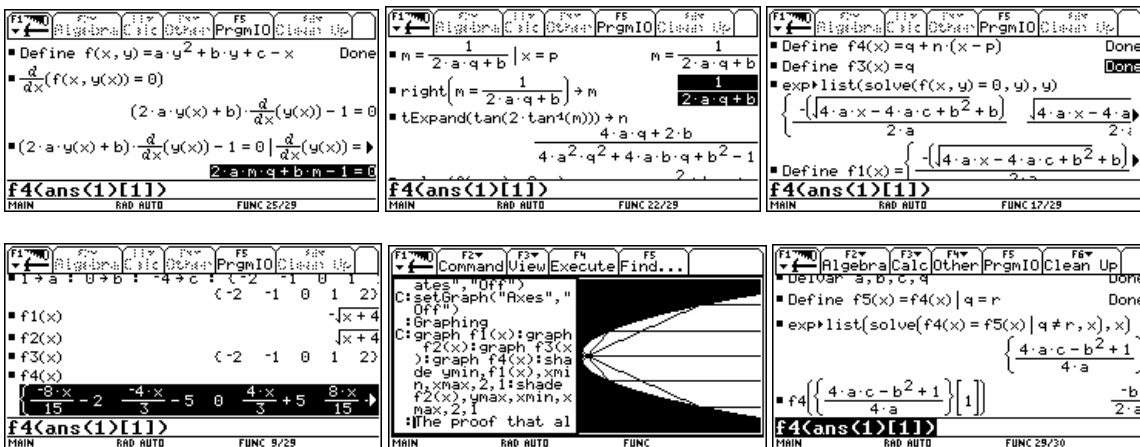


Figure 4 (from left to right) - (a) implicit differentiation; (b) slope of reflected ray; (c) mirror curves; (d) review of functions; (e) graphical representation; (f) proof.

Clearly, it was shown (and the graphical observation confirmed) that the reflected rays from the series of parallel incident rays pass through the focus of a parabola. Next we will investigate if the spherical mirror has the same property.

```

F1 Command View Execute Find...
: Define the function and use the implicit
: differentiation to find the slope of
: the tangent line m at the point (p,q)
: on the curve
C: Define f(x,y)=sqrt(c^2-y^2)+x
C: d(f(x,y(x)))=0,x
C: ans(1) d(y(x),x)=m and y(x)=q
C: solve(ans(1),m)
C: ans(1) |x=p
C: right(ans(1))>m
: Finding the slope of the reflected ray
: and the equation of reflected ray

...
: Selecting values for parameters of a m
: rror (if necessary) and the family of
: incident rays
C: 2+c: (-1.5, -1, -.5, 0, .5, 1, 1.5)+q
: Review functions prior to graphing
C: f1(x)
C: f2(x)
C: f3(x)
C: f4(x)
: Window and format settings
C: 2>xmin: -3>xmax: 3>ymin: -3>ymax
C: setGraph("Coordinates", "Off")
C: setGraph("Axes", "Off")
: Graphing
C: graph f1(x):graph f2(x):graph f3(x):gr
: aph f4(x):shade ymin,f2(x),xmin,xmax,2
: :ishade f1(x),ymax,xmin,xmax,2,1
: :switch to home screen and make correct
: ins if necessary
MAIN RAD AUTO FUNC

```

Figure 5 - Part II - parts of script

The script is the same as in Part I with the following changes:

- Instead of parabola, the mirror curve is the semicircle $x^2 + y^2 = c^2, x \leq 0$.
- The radius of the circle is 2 and the series of incoming rays is adjusted accordingly.
- After reviewing functions $f_1(x)$ and $f_2(x)$, prior to graphing, make necessary adjustments: First, $f_1(x)$ that was defined as when($x^2-c^2 \div 0$ and $x \div 0$, $\partial(c^2-x^2)$) should be redefined by typing on the Home screen when($x^2-c^2 \div 0$ and $x \div 0$, $\partial(c^2-x^2)$, undef))STO>f1(x). Same correction for $f_2(x)$: redefine it as when($x^2-c^2 \div 0$ and $x \div 0$, $\partial(c^2-x^2)$, undef))STO>f2(x).
- Shading needs to be reversed.

The following calculator screens illustrate the above script. Again, instead of showing a split screen (Text and Home screens), only the full Home screen is shown.

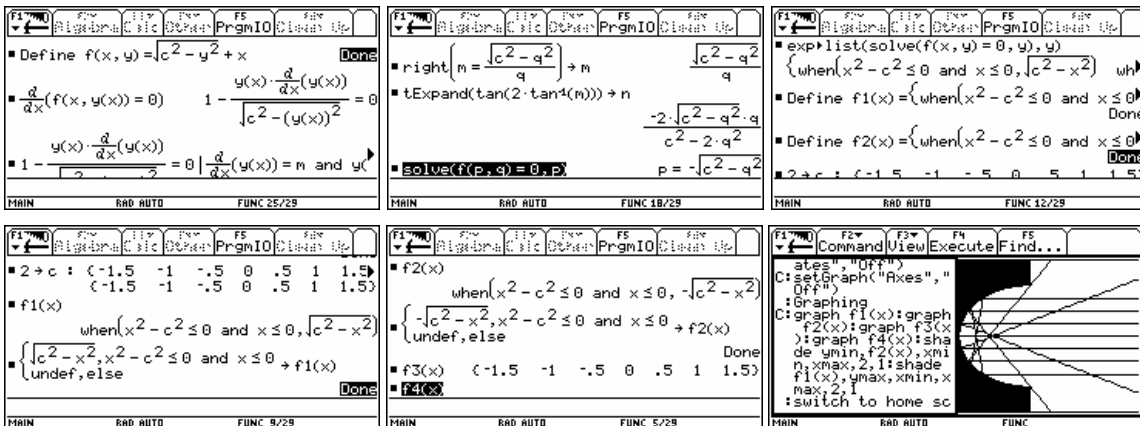


Figure 6 (from left to right) - (a) implicit differentiation; (b) slope of reflected ray; (c) mirror curves; (d)-(e) review of functions and redefining mirror curves; (f) graphical representation

Clearly, spherical mirror does not have the same property as parabolic. So the question remains: Is the parabolic mirror unique or do other surfaces exist that reflect a series of parallel rays into a single point?

```

F1 Command View Execute Find...
Derivation and solving differential equation whose solution is the curve with a property that all reflected rays from the series of horizontal incident rays pass through the origin
C:ClrHome:ClrGraph
C:SetMode("Split Screen","Left-Right"
Split 1 App,"Text Editor","Split 2 App"
"Home"))
C:DelVar x,y,n,c,d,t,u
y' - slope of a tangent line to the curve f(x,y)=0 at the point (x,y); n - slope of a reflected ray from the curve; y=nx - reflected ray that passes through (0,0) generates diff. eq. d.
C:tExpand(tan(2*atan(y')))+n
C:y=n*x
C:ans(1)+d
Solve d [it is homogenous diff. eq.]
C:deSolve(d,x,y)
Diff. eq. d is also Lagrange diff. eq. Alternative way of solving it leads to the solution in parametric form. Substitute y=t(x) and differentiate the whole equation with respect to x.
C:d|y=t(x) and y=n(x)
C:(ans(1),x)
Replace dt/dx as 1/(dx/dt) and solve the linear diff. eq. for x(t)
C:ans(1)|d(y(x),x)=t and d(t(x),x)=1/x' and t(x)=t
C:deSolve(ans(1),t,x)
the general solution in the parametric form x=u(t) and y=v(t), where u(t) is obtained above and v(t) follows from the eq. d where x is replaced by u(t) and y' by t.
C:ans(1)|@3=c
C:right(ans(1))+u(t)
C:d|x=u(t) and y=t
C:right(ans(1))+v(t)
Graphical representation
C:setGraph("Axes","On")
C:setMode("Graph","Parametric")
C:{-3,-2,-1,1,2,3}+c
C:ZoomStd
C:-10+tmIn:10+tmAx
C:Graph u(t),v(t),t
In our case it is possible to eliminate parameter t to obtain
C:DelVar c
C:solve(y=v(t),t)
C:x=u(t)|t=right(ans(1))
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Figure 7 – Part III script

- The slope of the tangent line to a smooth curve $f(x,y)=0$ at the point (x,y) is $m = y'$. Then the slope of the reflected ray at the same point is $n = \tan(2 \arctan y') = \frac{2y'}{1 - y'^2}$.
- The reflected ray that passes through the origin has an equation $y = nx$, which leads to the differential equation $y = \frac{2xy'}{1 - y'^2}$. Solving for y' leads to two homogenous differential equations that can be solved.
- The differential equation can be classified as a special case of Lagrange differential equation $y = xf(y') + g(y')$ whose general solution can be expressed in parametric form.
- Replacing y' with $t(x)$, differentiating with respect to x , leads to the linear first-order differential equation with unknown $x(t)$.
- Solving the obtained equation we find $x = u(t)$ while $y = v(t)$ follows from the original differential equation after the substitutions $y' = t$ and $x = u(t)$.

The following screens are obtained by executing the above script.

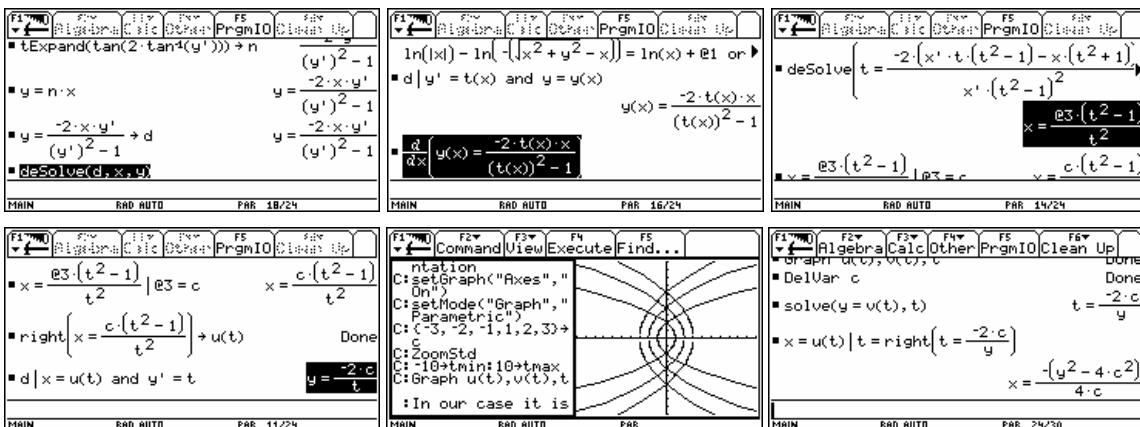


Figure 8 - (from left to right) - (a) derivation of diff. eq.; (b)-(c) solution using DE Solver and solving Lagrange equation; (d) parametric solution; (e) graphical representation; (f) equation of a parabola

Therefore, the parabola is the only curve that exhibits the unique reflection property: reflected rays from the series of parallel incident rays all pass through the same point.

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Appendix – Solving Lagrange and Clairaut Differential Equations

```

F1 Command View Execute Find...
: Lagrange diff. eq. y=xf(y')+g(y') where
: f(x)=x^2
C: DelVar d,f,g,x,y,t,u,v
C: ClrHome:ClrGraph
C: setMode("Split Screen","Left-Right","
: Split 1 App","Text Editor","Split 2 Ap
: p","Home")
C: Define f(x)=x^2
C: Define g(x)=x^3
: Differential equation
C: y=f(y)*x+g(y)
C: ans(1)->d
: Substitution, differentiation, and lin
: ear diff. eq. with unknown x(t)
C: ans(1)|y=y(x) and y'=t(x)
C: d(ans(1),x)
C: ans(1)|d(y(x),x)=t and d(t(x),x)=1/x'
: and
: Parametric equations
C: deSolve(ans(1),t,x)
C: right(ans(1)|@i=c)+u(t)
C: right(dx=u(t) and y'=t)->v(t)
C: x=u(t)
C: y=v(t)
: Graphing
C: [-2,-1,0,1,2]->c
C: setGraph("Axes","On")
C: setMode("Graph","Parametric")
C: ZoomStd
C: [-10+tmn:10+tmx]
C: Graph u(t),v(t),t
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```

Figure 9 – TI92 Plus script – Solving Lagrange differential equation

Lagrange Differential Equation:

- $y = xf(y') + g(y')$, $f(u) \neq u$ and $g(u)$ are differentiable functions.
- Clearly, the uniqueness theorem is not satisfied.
- Replace $y' = t$ and differentiate with respect to x :

$$y = xf(t) + g(t),$$

$$\frac{dy}{dx} = f(t) + xf'(t) \frac{dt}{dx} + g'(t) \frac{dt}{dx},$$

$$(t - f(t)) \frac{dx}{dt} - xf'(t) = g'(t).$$

- The above equation is a linear first order differential equation for $x(t)$. Solving it we find $x = u(t)$, and $y = u(t)f(t) + g(t)$.

```

F1 Command View Execute Find...
: Clairaut diff. eq. y=xy'+g(y')
C: DelVar c,d,dd,g,x,y,z,t,u,v
C: ClrHome:ClrGraph
C: setMode("Split Screen","Left-Right","
: Split 1 App","Text Editor","Split 2 Ap
: p","Home")
C: Define g(x)=x^2
: Differential equation
C: y=y'*x+g(y')
C: ans(1)->d
: Substitution, differentiation, and lin
: ear diff. eq. with unknown t(x)
C: ans(1)|y=y(x) and y'=t(x)
C: d(ans(1),x)
C: ans(1)|d(y(x),x)=t and d(t(x),x)=t' an
: d t(x)=t
C: ans(1)->dd
: General solution
C: deSolve(dd,x,t)
C: x*c+g(c)->z(x)
C: y=z(x)
: Singular solution
C: solve(dd,x)
C: right(ans(1))->u(t)
C: right(dly'=t and x=u(t))->v(t)
C: x=u(t)
C: y=v(t)
: Graphing
C: [-2,-1,0,1,2]->c
C: setGraph("Axes","On")
C: setMode("Graph","Parametric")
C: ZoomStd
C: [-10+tmn:10+tmx]
: Graphing general solution
C: y=z(x)
C: Graph t,z(t),t
: Graphing singular solution
C: Graph u(t),v(t),t
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Figure 10 - TI92 Plus script – Solving Clairaut differential equation

Clairaut Differential Equation:

- $y = xy' + g(y')$, $g(u)$ is a differentiable function.
- Clearly, the uniqueness theorem is not satisfied.
- Replace $y' = t$ and then differentiate with respect to x :

$$y = x \cdot t + g(t),$$

$$\frac{dy}{dx} = t + x \frac{dt}{dx} + g'(t) \frac{dt}{dx},$$

$$(x + g'(t)) \frac{dt}{dx} = 0.$$

- $\frac{dt}{dx} = 0$ implies $t = c$ (c – arbitrary constant) so $y = cx + g(c)$ is the general solution.
- The singular solution is given in parametric form $x = -g'(t)$, $y = -g'(t)f(t) + g(t)$ and it represents the envelope of the general solution.