Let the Pictures Tell the Story:
Visualizations for Mathematics Instruction

David R. Hill
Mathematics Department
Temple University
Philadelphia, PA 19122
hill@math.temple.edu

Lila F. Roberts
Mathematics and CSc Dept.
Georgia Southern University
Statesboro, GA 30460
lroberts@gasou.edu

Introduction

Few people would argue that a carefully chosen picture can convey some types of information in a more efficient way than the spoken or written word. Visual presentations grab the attention of learners, spawn mental snapshots of ideas, and can aid in the retention of information. Instructional presentations of this type are especially effective when the student's interpretation of the visualization is guided by an instructor in a classroom setting.

Many topics in mathematics become more meaningful when they are introduced using pictures or animations. With today's technology, it is relatively easy to develop visual materials that tell a story or illustrate ideas and relationships. Visual instructional components can be used as effective teaching tools by incorporating such features into an instructor-narrated demonstration.

In this paper we will discuss some effective visualizations that have been developed for *Demos with Positive Impact*, an NSF project to build a web-based collection of instructional demonstrations for teaching mathematics. In addition we will provide an overview of the techniques employed to develop the visualizations.

Visualization in Mathematics

"An idea. A tool. A change. Improvement. Educators today are bombarded with new approaches to teaching and learning, but one thing seems clear: nothing will change unless these ideas are brought to the frontlines of the classroom" [1]. Our view of the classroom is changing due to availability of technology and technological innovations such as the Internet and it will continue to evolve in ways that we cannot foresee. However, there is one constant that must be present for any type of classroom to be effective.

> **We must have instruction that conveys the fundamental ideas within the subject content in ways that address a variety of learning styles, utilizes visualization and imagery when appropriate, provides links to connect topics, and actively engages learners.**

There is tremendous power in using visualization to teach mathematics. The importance of visualization in general knowledge acquisition and problem solving processes has been well documented in educational studies. Looking at a picture, viewing an animation or video clip, and using a simulation are all virtual realities that have demonstrated the power to positively impact learning. Each of these virtual realities can be used as a vehicle of thinking to understand processes, concepts, and behaviors [2].

Empirical research tells us that students' construction of a representation of a mathematical object is based on the use of several symbolic representations. "Learning can be achieved through the translation between representations at different levels of abstraction" [3]. Visualization can be seen as providing relevant representations to assist the learner in carrying out the translation of abstract representations to ones which are less abstract. Thus visualization can bring valuable insight to students, particularly those who have difficulty with abstract mathematical objects. Visualization is a key to understanding a variety of complex mathematical topics [4].

The seminal collection of articles in [5] provides a set of ideas and a resource of examples for visualization in mathematics that has guided a number of investigations over the past decade as the power of calculators and computer generated imagery dramatically increased. In our development of visual tools in the NSF project *Demos with Positive Impact* we have been guided by a number of items expressed in [6]. In particular,

- we use the term visualization to describe the process of producing and using geometrical or graphical representations of mathematical concepts, principles, or problems,
- we subscribe to the notion that "in mathematics, visualization is not an end in itself but a means toward an end, which is understanding,"
- we believe that mathematical visualization should aid in the development of "intuition through pictures formed in the mind's eye," and
- as instructors, we think it is important to realize that "vision is not visualization; to see is not necessarily to understand."

With these guidelines in mind we have developed animations to accompany a number of the demos in our collection *Demos with Positive Impact* which can be found at URL http://www.mathdemos.org. When we constructed accompanying visualization tools we tried to incorporate basic features that made the visual component easy to use, adaptable to a variety of levels, and flexible for use in a variety of classroom settings. For example,

- the animations are easily displayed with commonly available tools on both PCs and Macs;
- we use color and keep frame layouts simple;
- a step-wise progression of steps shows the development of the topic;

- the speed at which visualization components progress is slow enough for students to simultaneously assimilate an instructor's dialogue, (if necessary, demos can be re-run with little effort);
- where appropriate we suggest outlines, often incorporating suggestions from the demo contributor, on how to structure dialogue to accompany the animation;
- we strongly suggest that software with stop-start features (such as Quicktime) be used so that student questions or remarks along the way can be addressed. This also provides an easy way to repeat parts of the animation.

Ideally, a student should be able to run the animation on his or her own. A replay of the demo can refresh or even provide a more detailed examination of the concepts discussed during the class. Hence, any technology platform employed by the demo should be accessible to the student; unfortunately, this is not always possible or practical. To help to overcome this difficulty, we have tried to develop browser capable versions of animations (animated gif, Java applet, or Javascript).

Visualization Tools for Calculus

The literature confirms that visualization is of key importance in Calculus. "At the beginning college level, visualization is a big part of understanding" [7]. In [8] it is noted,

> Of all undergraduate mathematics courses, none offers more interesting and varied opportunities for visualization than calculus. Most of the concepts and many problems of calculus can be represented graphically. In many cases, however, these are little more than decorations. In selected cases, diagrams may be used directly as a tool in problem solving, but considering the calculus course as a whole, geometrical reasoning is used inconsistently at best, and the role of visual thinking is not seriously addressed. Few of the examples or problems are designed to develop the student's ability to represent or solve problems graphically. In calculus, we face the paradox that the importance of graphics is widely recognized, but the potential inherent in visualization has not been achieved.

Since the preceding rather bleak assessment was made over a decade ago there has been progress, especially through the calculus reform initiatives that have been proposed, tested, used, refined, and which continue to foster controversy.

The project *Demos with Positive Impact* contains demonstrations of a variety of topics from pre-calculus, calculus, and post calculus. Here we provide a brief description of collections of animations that address topics in calculus. Two of the areas on which we have focused are volumes of solids and application (word) problems.

The objective of *Visualizations for Volumes of Solids in Calculus* is to help students understand the concepts that motivate the elements of computation of volumes of solids of revolution. Rather than introduce volumes of solids of revolution by a purely formula-driven approach, these demos provide the opportunity for visualization of the basic approximating elements that lead to the standard calculus expressions that, when computed, give the desired volumes.

A gallery of visualization tools has been developed for teaching volumes in Calculus. Examples are included for the **disk method, method of shells, washer method, and cross section method**. We provide a variety of animations for some common examples. For example in the method of shells two formats, animated gif and mov are provided with Regions Bounded Between a Curve and an Axis on an Interval [a, b] and Regions Bounded Between Two Curves. A particular example is the region **R** in the first quadrant bounded between $y = x^2$ and $y = \sqrt{x}$. The images in Fig. 1 are a sample of the kind of views of the generation of the solid that are contained in the animation.

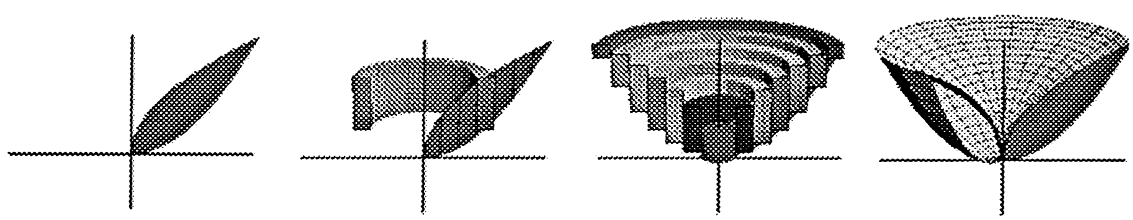

Figure 1.

The animation also shows the solid generated by revolving **R** about the y-axis. Included with the demo is a step-by-step narrative script of the displays in the animations. This collection is included in the MAA's Digital Classroom Resources; see [9]. You can access the 'Collection of Volume' animations at the URL in [10].

Word problems that illustrate applications in calculus are addressed by three collections, *Constructing Equations from Word Problems* (see [11]), *Demos for Related Rate Problems* (see [12]), and *Demos for Optimization Problems* (under construction). In these collections we include animations, some interactive MATLAB files, and some interactive student-developed java applets. A sample of the demos addressed for related rates is pictured in Fig. 2.

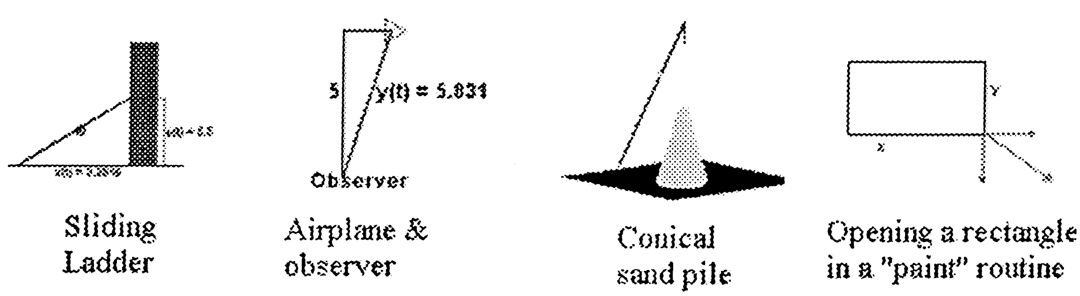

Figure 2.

Summary...and an Invitation

Visualizations in the form of animations are powerful tools to stimulate students to think about mathematical concepts and problem solving. Our project *Demos with Positive Impact* has been recognized for contributions to visualization by the MAA's Digital Classroom Resource program and chosen for the Eisenhower National Clearinghouse October 2002 Digital Dozen. We had favorable reviews by The College Board's AP CENTRAL. We invite faculty to visit our website, become a user of existing materials, a contributor of demos that you have found really help your students, or a reviewer of materials.

Acknowledgement. Partial support for this work was provided by the National Science Foundation's Course, Curriculum and Laboratory Improvement Program under grant DUE-9952306.

REFERENCES

1. R. Soine, 'Technology in the Trenches: Improving the Quality of Instruction,' ***T.H.E. Journal***, Vol. 26, No. 5, pp. 51-52, 1998.
2. National Science Foundation, Directorate for Education and Human Resources (EHR), Division of Research, Evaluation and Communication (REC), Annual Impact Report, Fiscal Year 1996, 'What We Learned in FY 1996', http://www.ehr.nsf.gov/ehr/rec/recfy96/sect4b.htm
3. Computeralgebra in Lehre, Ausbildung und Weiterbildung, 'Visualization: Courseware for Mathematics Education', http://www.gwdg.de/~cais/CAR/CAR26/node13.html
4. North American Chapter of the International Group for the Psychology of Mathematics Education, Working Group on Representations and Mathematics Visualization (1998-2001), 'Working Group on Representations and Mathematics Visualization PMENA XX (North Carolina, 1998)', Fernando Hitt, Center for Research and Advanced Studies (Cinvestav), Mexico, http://www.matedu.cinvestav.mx/index3.html
5. *Visualization in Teaching and Learning Mathematics*, Walter Zimmerman and Steve Cunningham, editors, The Mathematical Association of America, 1991 (MAA Notes Number 19)
6. 'Editor's Introduction: What is Mathematical Visualization?', in *Visualization in Teaching and Learning Mathematics*, Walter Zimmerman and Steve Cunningham, editors, The Mathematical Association of America, 1991 (MAA Notes Number 19)
7. 'Visualization and Calculus Reform', Deborah Hughes Hallet, in *Visualization in Teaching and Learning Mathematics*, Walter Zimmerman and Steve Cunningham, editors, The Mathematical Association of America, 1991 (MAA Notes Number 19)
8. 'Visual Thinking in Calculus, Walter Zimmerman, in *Visualization in Teaching and Learning Mathematics*, Walter Zimmerman and Steve Cunningham, editors, The Mathematical Association of America, 1991 (MAA Notes Number 19)
9. http://www.mathdl.org/dcr_index.html
10. http://www.cs.gasou.edu/faculty/demos/solids/
11. http://astro.temple.edu/~dhill001/wordproblemeqn/wordproblemeqn.html
12. http://astro.ocis.temple.edu/~dhill001/relatedrates/relatedrates.html