SYSTEMATIC FORMULATION OF MATHEMATICAL WORD PROBLEMS INVOLVING UNITS

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Introduction

Systematic methods that assign meaning to quantity expressions in word problems can significantly impact students' abilities to formulate mathematical models and calculate solutions. Unfortunately, ambiguity and confusion exists in current literature primarily because of the absence of precise definitions and systematic methods that formally assign meaning to quantities in quantitative problems. For example, does "3 = 1"? Do "3 feet = 1 yard"? Do "3 feet of board = 1 yard of yarn"? Can we add 2 feet and 3 pounds? Can we add 4 apples and 5 oranges? What do we mean when we define a variable using an equation such as "x = length"?

This ambiguity also leads to conflicting presentations of some elementary concepts taught in mathematics. For example, consider the definition of "ratio" from two typical high school textbooks. On page 190 of the textbook <u>Algebra-Concepts and Applications</u> published by Glencoe/McGraw Hill the authors state that "a rate is a ratio of two measurements having different units of measure." But, on page 109 of the text book <u>Algebra 1-An Integrated Approach</u> published by Heath/Houghton Mifflin the authors state that "a ratio compares two quantities measured in the same units". The use of these examples should not be interpreted negatively on these excellent authors; instead, it reflects the current state of the art.

Such issues occur because current literature does not carefully define and combine the concepts of units and dimensions into precise meanings associated with quantities in quantitative problems. As an illustration, after the ratio definition on page 109, the Heath textbook gives the following illustration: "win-loss ration = games won / games lost = 10 games / 6 games = 5 / 3". Notice how this typical example illustrates the widely accepted vague mixing of quantities and information about those quantities. What is the precise mathematical meaning of such expressions and equations?

Also on page 190, the Glencoe textbook defines "dimensional analysis" as "the process of carrying units throughout the computation." Other textbooks refer to this process as "factoring analysis", "units analysis", etc. But, this widely used practice of canceling units is still implemented in a vague manner primarily as labels on the quantities.

Furthermore, especially in the physical sciences, the phrase "dimensional analysis" refers to a completely different process in which the quantities are replaced by base dimension symbols (not the same as "units") and then algebraically simplified to reduce the number of independent dimensions or verify the validity of relationships. Some authors (like Naddor, Vignaux) have used brackets to indicate dimensional information associated with quantities, using expressions such as "F [ML/S^2]" to indicate that the quantity "force" has a dimensional expression of mass times length divided by seconds squared. The bracketed expression is used as a parenthetical label indicating that the dimension expression can replace the F in a formula such as "F=ma" to verify and manipulate dimensional structure. Naddor used symbols such as "$" and "Q" to represent cost and quantity dimensions.

The formal notational structure proposed by National Institute of Standards and Technology (NIST) in the <u>Guide for the Use of the International System of Units (SI)</u> falls short of being comprehensive as it suggests the following use of notation for expressing the values of quantities:

> "the value of quantity A can be written as A = {A} [A], where {A} is the numerical value of A when the value of A is expressed in the unit [A]. The numerical value can therefore be written as {A} = A / [A], which is a convenient form for use in figures and tables. Thus to eliminate the possibility of misunderstanding, an axis of a graph or the heading of a column of a table can be labeled "t/°C" instead of "t(°C)" or "Temperature (°C)". Similarly, an axis or column heading can be labeled "E/(V/m)" instead of "E(V/m)" or "Electric field strength (V/m)" (Section 7.1 of the Guide)

This notation only combines the numeric value of a quantity to the symbol and unit; it does not involve the dimension. In fact, any attaching or mixing of information (including dimension information) with units is explicitly stated as unacceptable (Section 7.4 and 7.5 of the Guide), most likely because they found no current literature that provided a consistent method of doing this.

In software applications, unit labels are often used in specific computational contexts. For example, units are frequently to determine unit conversions. More specifically, graphical design software applications (such as AutoCad and TurboCAD) provide methods to input length units associated with specific objects and allow the user to apply unit conversions over a collection of objects. Such software generally uses the term "dimension" to refer to the "length" dimension of various linear measurements on a two or three-dimensional diagram. Project management software applications (such as Microsoft Project and Primevera) and some of the graphical design software applications provides methods to access databases of cost and time information to determine total costs and time constraints of collections of objects and events. Mathematics solving, optimizing, and graphing software (such as LiveMath, Maple, MathCAD, Mathematica, MATLAB, OptiMax, TK Solver, etc.) employ methods of tracking units to verify the validity of multiplying quantities. Geographical information systems and other mapping software support different unit scales. Modeling and simulation software applications (such as SansGUI, SimCAD and Simulink) also provide modules for unit conversions. Specialized calculators (such as Measure Master Classic, NautiCalc Plus, ProjectCalc and Real Estate Master) allow the user to enter specific types of related units (even using special keys) and prompt the user with unit labels during the inputting of numeric information into preset formulas (again accessible by special keys).

In all these examples, the user is still required to enter the mathematical expressions in the same traditional way of entering quantities, operators, and mathematical functions. Unified Mathematics introduces the novel idea of entering the meaning (using a formally defined combination of units and dimensions) associated with the quantities and then having the system determine operators and mathematical functions for the model.

Most text and trade books devoted specifically to methods of solving quantitative problems devote themselves to "types" of problems (rate problems, percent problems, volume problems, unit conversion problems, etc.). Even the recent patent "System and methods for searching for and delivering solutions to specific problems and problem types" (US Patent No. 6,413,100) finds solutions to word problems using this traditional approach. Unfortunately, these traditional approaches still remain ambiguous when dealing with meaning.

<u>Orientation to Unified Mathematics</u>

Unified Mathematics consists of proprietary systems and methods developed by Multispan Productions, Inc. that allow the meaning of the quantities in a quantitative problem to control the modeling process; and as a consequence pave the way for a new generation of calculator applications where the user inputs the meaning of the problem and then the software application formulates the model and calculates solutions. This allows students to focus on critical thinking with the meaning of quantities rather than getting overwhelmed by the mechanical operations and solution processes that computers can easily provide. It satisfies the need for current educational approaches to provide a comprehensive framework to formulate mathematics and science problems, so that students will not become unduly frustrated with their ability to understand the role that mathematics plays in science. Indeed, Unified Mathematics opens up the opportunity to have a centralized database of meanings that define commonly used dimensions used by this new generation of software applications when formulating quantitative problems.

Briefly, Unified Mathematics attaches meaning by generalizing the concept of a <u>dimension</u> D to a property function of a thing and qualifies a <u>unit</u> u with the dimension that it measures using the symbolic notation "[u~D]". This <u>qualified unit</u> "[u~D]"is then attached to the quantity q resulting in a <u>unified quantity</u> having the symbolic notation "q [u~ D]". By breaking the recommended unacceptability of attaching information to units, this significantly extends the NIST notation where u corresponds to [A], q corresponds to A, and D is attached information that qualifies the unit.

The qualified unit [u~D] is not just a label (for quantities, tables, and graphs) as done in current literature, but the open bracket "[", close bracket "]" and tilde "~" (read "of") are mathematical operators; and the unit u, the dimension D, and the qualified unit [u~D] become symbols that can participate in algebraic manipulations along with the quantities associated with them. This novel approach goes beyond the loosely defined "factoring" or "substitution" processes used in current literature where the units participate in simple cancellation or substitution processes only.

Unified Mathematics captures the complete meaning of quantities in contrast to other approaches such as that proposed by the National Institute of Standards and Technology:

Unified Mathematics	NIST
π [m~circumference]/[m~diameter]	π (dimensionless)
c ([m~distance in vacuum]/[s~time])	c /(m/s)
t [˚C ~ Temperature]	t/˚C
E [V/m ~ Electric field strength]	E/(V/m)

Indeed, this novel concept of unified quantities allow us to clearly define the meaning of functional relationships. Consider, for example, a simple annual interest calculation. Consider the usual (ambiguous) approach using financial quantities: "I = P * r * t, where I is the interest, P is the principle, r is the rate, and t is the time." Instead, the methods of Unified Mathematics construct a <u>unified relationship</u> of unified quantities as follows: "I [dol~interest] = P [dol~principle] * r ([dol~interest]/([dol~principle]*[yr~time])) * t [yr~time]". Furthermore, this approach clearly defines the meaning of a given quantity; for example, in the above unified relationship, the rate is clearly defined as an annual interest rate (not a monthly rate, etc.). It is generally known that units determine the constants that appear in relationships, and so unified relationships with their particular constants become self documenting.

To illustrate how we can manipulate the brackets as symbols, consider substituting "12*month" for "yr" in the unified quantity "t [yr~time]" the methods of Unified Mathematics provide a systematic way to algebraically "pull" the constant 12 out of the bracket operator to the front of the unified term resulting in "t [yr~time] = t [12*month ~time] = (12 * t) [yr~time]" which yields: "I [dol~interest] = 12* P [dol~principle] * r ([dol~interest]/([dol~principle]*[yr~time]))* t [month~time]".

The systems and methods of Unified Mathematics introduces rules such as the addition rule: "q_1 [u~D_1] + q_2 [u~D_2] = (q_1 + q_2) [u~(D_1+D_2)]". Furthermore, these systems and methods introduce on a new unit, designated "ins" for "instance" with a corresponding dimension of "occurrence". This new approach resolves the dilemma of adding apples and oranges, since in the addition rule the unit "ins" would be the common unit in the expression: "10 [ins~App] + 25 [ins~Ora] = (10+25) [ins~(App+Ora)]", where, for example, the symbol "App" abbreviates the dimension "occurrences of apples" and we read the phrase "ins~App" as "instances of occurrences of apples" or for brevity (but not ambiguity), "instances of apples". Unified Mathematics also proposes an extension of the international system of units (SI) by adding two new base units, the <u>dollar</u> (dol) to measure the dimension “monetary value” of a thing. These new base units allows Unified Mathematics to apply to business and statistics problems.

Bibliography

BHAGWAT, AMIT, “Applying Dimensional Analysis to Business Intelligence Systems”, Journal of Conceptual Modeling, 2002, Issue: 24.

CUMMINS, JERRY, etal. <u>Algebra: Concepts and Applications</u>, 2001, p. 190, Glencoe/McGraw-Hill, Columbus, OH

DAVID HALLIDAY, ROBERT RESNICK, <u>Fundamentals of Physics</u>, 1974, pp. 48-51, John Wiley and Sons, Inc., New York, NY.

DE JONG, F. J., “Dimensional Analysis for Economists, Contributions to Economic Analysis”, 1967, Vol. 50, North-Holland Publishers, Amsterdam.

JOHN SAXON, <u>Algebra II, An Incremental Development</u>, 1985, pp. 142-145, Grassdale Publishers, Inc. Norman, OK.

KASPRZAK, WACLAW, BERTOLD LYSIK AND MAREK RYBACZUK, <u>Dimensional Analysis in the Identification of Mathematical Models</u>, 1990, World Scientific Publishing Co., Inc., Teaneck, NJ.

LARSON, ROLAND E., TIMOTHY KANOLD AND LEE STIFF, <u>Algebra 1: An Integrated Approach</u>, 1997, p. 109, D.C. Heath and Company, Lexington, MA.

NADDOR, ELIEZER, <u>Inventory Systems,</u>1966, John Wiley & Sons, New York, NY.

NADDOR, ELIEZER, “Dimensions in Operations Research”, Operations Research, 1966, pp. 508-514, vol. 14.

POLYA, G., <u>How to Solve It – A New Aspect of Mathematical Methods, 2-nd Edition</u>, 1998, Princeton University Press, NJ.

SZEKERES, PETER, The Mathematical Foundations of Dimensional Analysis and the Question of Fundamental Units, International Journal Theoretical Physical, 1978, pp. 957-974, vol. 17, no. 12.

TAYLOR, BARRY N., Guide for the Use of the International System of Units (SI), NIST Special Publication 811, 1995 Edition, Gaithersburg, MD

VIGNAUX, G. A., Dimensional Analysis in Operations Research. New Zealand Operations Research, 1986, pp. 81-92, vol. 14.