Using the TI 89/92 in Number Theory and Abstract Algebra

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A lot has been said about the use of the graphing capabilities of calculators in mathematics classes. This presentation, however, focuses on the use of the symbolic capabilities of the TI 89 and TI 92. The calculators' ability to do algebraic manipulations in symbolic form provides many opportunities for students to generate examples and explore patterns in higher level mathematics courses. In particular, this presentation looks at an activity for Number Theory.

Fibonacci and Lucas Numbers

Most mathematics students are familiar with the Fibonacci sequence, which begins with the "seeds" $F_1 = 1$, $F_2 = 1$ and later terms are defined in an iterative manner with $F_{n+2} = F_n + F_{n+1}$. The beginning of the sequence is

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

Many students may not be aware of the Lucas numbers, which form a sequence similar to the Fibonacci numbers. The only difference is in the seed numbers used to start the sequence; rather than 1 and 1, 1 and 3 are used. The same iterative definition is followed for subsequent terms: $L_{n+2} = L_n + L_{n+1}$. The beginning of the Lucas Numbers sequence is

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, \dots$$

Make a sequence by looking at the ratios of consecutive terms of the Fibonacci sequence:

$$\frac{F_{n+1}}{F_n}.$$

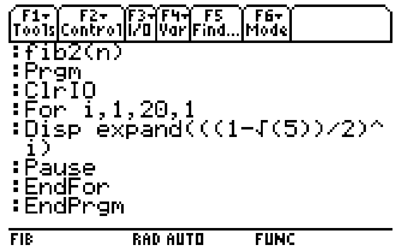
This sequence will converge to the Golden Ratio:

$$\frac{1+\sqrt{5}}{2}.$$

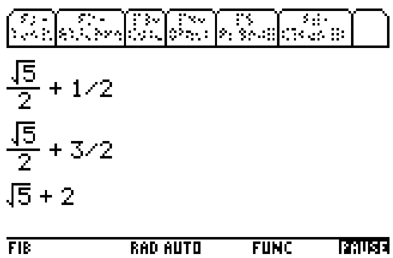
In addition, the similar sequence formed from ratios of consecutive terms of the Lucas numbers will also converge to the Golden Ratio. With this in mind, we will use the TI 89/92's *expand* command to investigate powers of the Golden Ratio and its algebraic conjugate.

$$\begin{aligned} a &= \frac{1+\sqrt{5}}{2} \\ b &= \frac{1-\sqrt{5}}{2} \end{aligned}$$

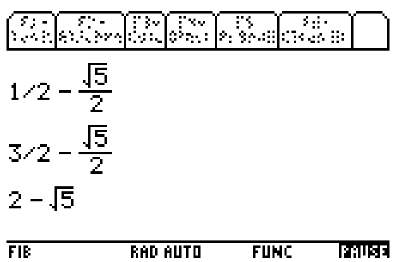
The following programs for the TI 89/92 will display the powers of a and b in exact form:



The first few outputs for the powers of a are



while the first few outputs for the powers of b are



The patterns will emerge when the results are put in the form

$$\frac{c \pm \sqrt{5}}{2}:$$

n	a^n	b^n
1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
2	$\frac{3+\sqrt{5}}{2}$	$\frac{3-\sqrt{5}}{2}$
3	$\frac{4+2\sqrt{5}}{2}$	$\frac{4-2\sqrt{5}}{2}$
4	$\frac{7+3\sqrt{5}}{2}$	$\frac{7-3\sqrt{5}}{2}$
5	$\frac{11+5\sqrt{5}}{2}$	$\frac{11-5\sqrt{5}}{2}$
6	$\frac{18+8\sqrt{5}}{2}$	$\frac{18-8\sqrt{5}}{2}$
7	$\frac{29+13\sqrt{5}}{2}$	$\frac{29-13\sqrt{5}}{2}$

From this, we can see that

$$\begin{aligned} a^n &= \frac{L_n + F_n\sqrt{5}}{2} \\ b^n &= \frac{L_n - F_n\sqrt{5}}{2}. \end{aligned}$$

As an exercise, this pattern can be proven for all $n \geq 1$ using induction.

From this, we can derive the Binet formulas for F_n and L_n,

$$\begin{aligned} F_n &= \frac{a^n - b_n}{a - b} \\ L_n &= a^n + b^n. \end{aligned}$$

As an extension, we can define a Fibonacci-type sequence S_n with the iterative definition

$$S_{n+2} = cS_n + dS_{n+1}$$

where c and d are arbitrary non-zero integers. In this general case, if we look at the sequence of ratios of the consecutive terms, we will also have a limit

$$\lim_{n\to\infty} \frac{S_{n+1}}{S_n} = \frac{d + \sqrt{d^2 + 4c}}{2}.$$

For example, choose $c = 1$, $d = 3$ and let the seeds be 1 and 1. The iterative definition is

$$S_{n+2} = S_n + 3S_{n+1}$$

and the first terms of the sequence are

$$1,\ 1,\ 4,\ 13,\ 43,\ 142,\ 469,\ \ldots$$

and the limit of the consecutive terms is

$$e = \frac{3 + \sqrt{13}}{2}.$$

Using the TI 89/92, we find that the consecutive powers of this limit are

n	e^n
1	$\frac{3+\sqrt{13}}{2}$
2	$\frac{11+3\sqrt{13}}{2}$
3	$\frac{36+10\sqrt{13}}{2}$
4	$\frac{119+33\sqrt{13}}{2}$
5	$\frac{393+109\sqrt{13}}{2}$

Thus the powers of e seem to fit into the pattern

$$e^n = \frac{T_n + U_n\sqrt{13}}{2}$$

where T_n and U_n are Fibonacci-type sequences that both fit satisfy the iterative definition of S_n.

Further Questions

1. Can the conjecture about T_n and U_n above be proven using an inductive argument?
2. Given integers c and d, expand the powers of

   $$\frac{d + \sqrt{d^2 + 4c}}{2}.$$

   Will these powers have coefficients that are Fibonacci-type sequences of the form $S_{n+2} = cS_n + dS_{n+1}$?
3. Can the seeds for the sequences you arrive at from the expansion of

   $$\frac{d + \sqrt{d^2 + 4c}}{2}$$

   be determined in terms of c and d?