A Spiral Approach to Problems in Linear Algebra: Computer Laboratories

B. Elenbogen, D. James & M. Lachance, University of Michigan – Dearborn

ABSTRACT: Interactive linear algebra laboratories at UM-D have the innovative feature that the presentation of each laboratory's focus topic spirals through three levels of increasingly sophistication: (1) basic, (2) realistic, (3) advanced. This increasing-levels-of-maturity approach provides students with the bridge to real-world application. It also allows the instructors to choose the level to which to expose their class.

INTRODUCTION

A linear algebra project at the University of Michigan – Dearborn has developed and is developing pedagogical materials for sophomore level linear algebra. Our University is a mid-sized (6,600 undergraduates) state-supported, commuter university located in metropolitan Detroit. A materials development project funded by a grant from the National Science Foundation is currently underway, and has as one of its primary components, the development of computer laboratories.

Most applications in linear algebra fall into one of eleven categories. For each category we have developed a laboratory which focuses on the features common to that category. All laboratories are heavily dependent on computer use.

1) Using Matrices as Data Storage Devices (A)
2) Flow-Type Problems (Ax=b)
3) Stochastic Matrices (Markov Processes) ($Ax_{n-1} = x_n$)
4) Steady State / Equilibrium Problems (Ax = x)
5) Network Problems: Adjacency Matrices (A^n)
6) Potentials: Incidence Matrices (A x = b – R y)
7) Measuring Proximity (A^TA and AA^T)
8) Curve Fitting: Overdetermined systems ($A^TAx = A^Tb$)
9) Computer Graphics ($Ax_{n-1} = x_n$)
10) Decoupling Problems: Eigenvalue and Eigenvector ($B^{-1}AB = D$)
11) Uses of the Singular Value Spectral Theorem

Each laboratory begins with a few straightforward examples illustrating the concept. Then a second level of maturity is introduced by revisiting the topic, this time addressing real-world complications, and making adjustments to the solution found at the first level. This maturing procedure continues until several real-world levels of practicality have been incorporated. Finally, the topic is revisited a third time, introducing yet more realism and presenting relevant special techniques. This level-of-maturity process allows students to learn how to refine a simple model so it will reflect a closer approach to reality. It also allows the instructor the freedom to choose the level to which he or she wishes to expose the class. And it provides advanced students with avenues for learning advanced material.

Each module concludes with a summary section advising the user how to decide whether a given problem will fit into the general format described in that module. The development of this skill, ignored in most texts and labs, may well be the one of greatest use to students after they leave the classroom. The goal of these summary sections is to help the student develop into an intelligent user of linear algebra, and to incorporate linear algebra techniques into his or her arsenal of problem solving tools.

The mode of the presentation is interactive: some parts of the module are presented in finished form, and for other parts students are instructed to work out well-defined parts of the development. At the end of each level are several relevant exercises.

In this paper we present an overview of two of the laboratories in our series.

Example 1: Laboratory on Flow-Type Problems.

The flow of quantities from one place to another can be recorded as a system of equations; such systems can then be expressed in matrix form as **Ax**=**b**. Flow can involve any kind of object, such as 1) the flow of passengers in an airport between terminals, or 2) the flow of water in a watershed area. Two more sophisticated types of flow are 3) a water flow system through pipes with cross-branchings, where the rate of flow can be regulated by valves, and 4) the flow of traffic along a grid of streets. Other flow problems can involve money, molecules or even such abstract concepts as length or time.

Level 1: (Unbranched Tree Diagrams: unique solution)

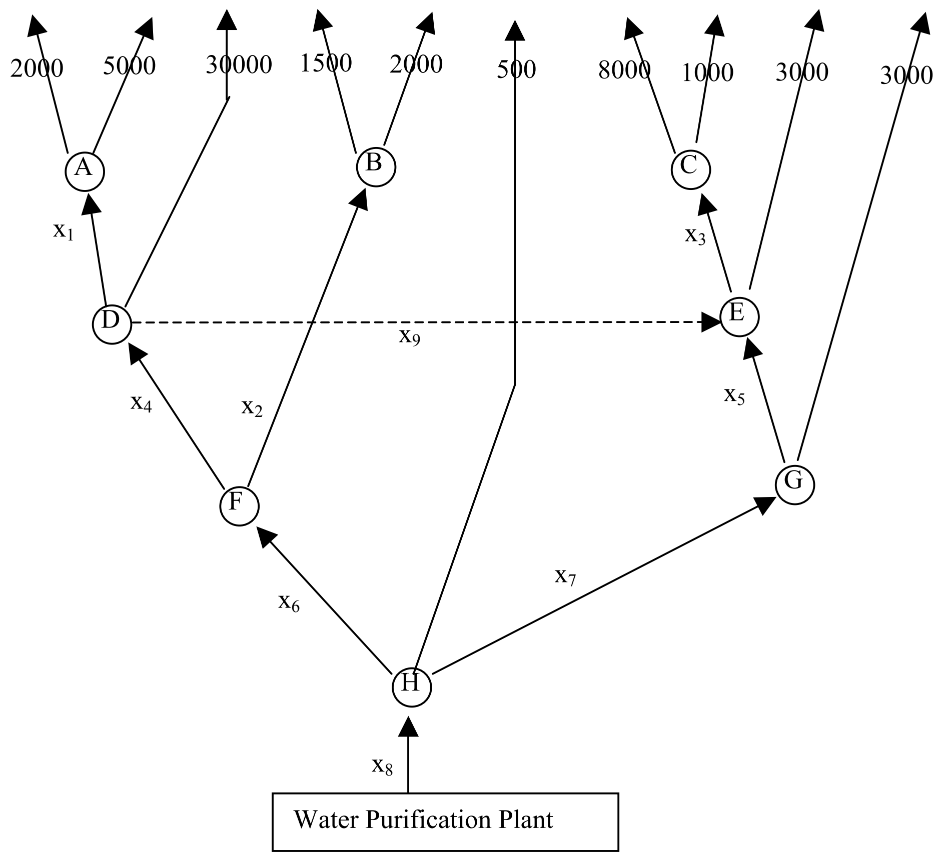
The simplest flows are those modeled by an unbranched tree diagram. The flow of water through a regional watershed corresponds to a tree the nodes of which are the points where two or more streams merge. In a second example, the newspaper distribution system, the flow is of published papers which all start at a single printing plant, and are distributed to regional substations, and then divided again and bundled to the delivery people, and then finally delivered to the customers. The nodes are the places subdivision occurs.

In both examples, there is a balancing at each node in the sense that the "total amount in" must equal the "total amount out". This balancing is the key to flow-type problems, since at each node, this balance gives an equation, and the resulting system of equations can be expressed as a matrix equation Ax = b, and solved.

Another mentioned is the distribution of supplies in a fast food chain, where the supplies flow out from national headquarters to regional headquarters, to the individual stores, to the customers. Interestingly, the payments for these supplies flow through the same tree diagram, but in the opposite direction.

Level 2: (Extension to Systems with Cross-Branching: many solutions)

A large water purification plant supplies the needs of the ten clients at the top of the diagram shown below. Initially the dotted x_9 pipeline is absent. The needs of all clients can be supplied so long as long as there is no blockage in any pipeline. But if a pipeline, say x_6, becomes clogged or contaminated, then up to half the clients would lose their water supply (assuming x_9 is not present.) The way to improve this situation is to install more pipes to allow multiple ways to move water from one node to another. For example, new pipe could be installed at x_9. After it is installed, if any pipe breaks, then at most two of the ten clients will be without water service, rather than five.



With the dotted pipeline installed, water can go from H to E in two different ways; we no longer have a simple tree, we have cross-branching.

The system of equations resulting from the balancing of "in" versus "out" at the nodes will be 8 equations in 9 unknowns. Since the number of variables exceeds the number of constraints (equations), we expect a parameter in our solution, and therefore expect infinitely many solutions. That turns out to be the case for this example, and students are instructed to carry out the details in this example.

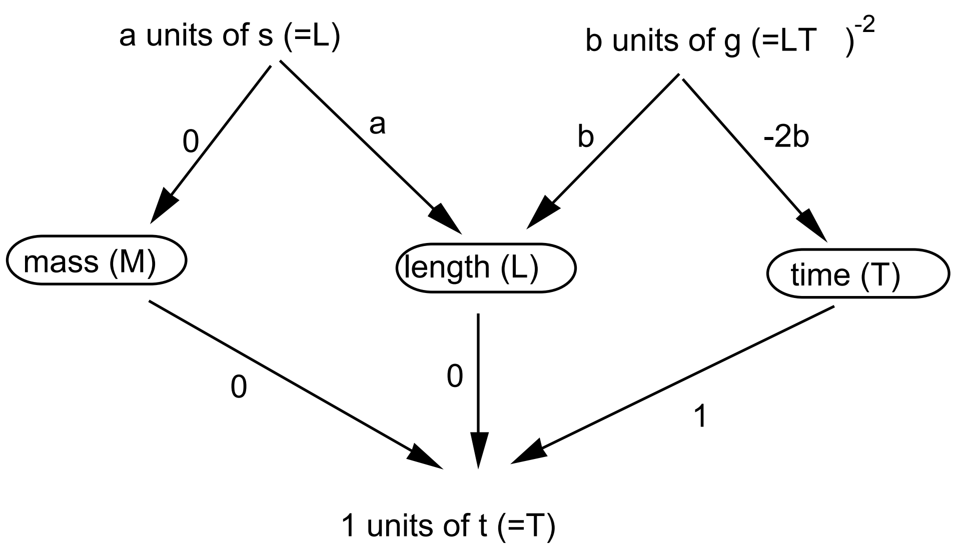
Level 3. Simultaneous Flow of several Species: X-Gates.

At the third level, a new type of node is introduced which has restricted entry. A type of modeling, more sophisticated than at the previous levels, can be achieved by the gluing together of several of these so-called X-gates into a complex, capable of handling real world problems involving sorting. Examples of quantities sorted are eggs according to size, airline passengers according to needs, and chemical compounds according to elements.

An example of an abstract application of X-gates involves determining a relationship between t, s, and g where t is the time it takes a body to fall, s is the distance it falls, and g is the acceleration due to gravity. The categories into which we sort are M (mass), L (length) and T (time). To begin we write

$$t = k\, s^a g^b \quad \text{(where k is a dimensionless constant.)}$$

where what we want to know are the values for a and b. The dimension of the left side is time alone, that is, $M^0L^0T^1$. On the right side, s is a length so it has dimension L, and g is acceleration so it has dimensions $L\,T^{-2}$. The dimensional counterpart to the equation above is therefore: $T = (L)^a (LT^{-2})^b = M^0 L^{a+b} T^{-2b}$. We have a three interconnected X-graphs, with three restricted-entry nodes: Mass (M), Length(L), and Time(T). Matching exponents of the left side of the equation (top of diagram) with the



right side (bottom), we have

$0 = 0$ (balance at the M-node)

$0 = a + b$(balance at the L-node)

$1 = -2b$ (balance at the T-node)

which (if the 0=0 identity is ignored) is a system of two equations in two unknowns, expressible in augmented matrix form as $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 1 \end{array}\right]$. Back substitution yields b = -1/2 and a = 1/2, so the equation is

$t = k\, s^{(1/2)}\, g^{(-1/2)},$ or in more familiar form, $t = k\sqrt{\frac{s}{g}}$.

We have found this formula without observing a single experiment, an impressive outcome achieved with unbelievable ease. Several more complicated dimensional analysis problems are presented for the student to solve.

Example 2: Stochastic Processes Laboratory

Level 1. At this basic level, several of the usual examples involving stochastic processes are presented: population movement, shifts in brand loyalty among consumers, rats in a maze. These are well enough known to need no further explanation in this paper.

Level 2. At this Level, students learn how to redefine the categorizing process to acquire more information from the analysis. Even more important is the study of the many situations in the real world which have a stochastic feel to them, but fail to be stochastic in some way. Level 2 presents ways to convert these nearly-stochastic problems into stochastic ones, by introducing new categories into which the misfits might fall. This can be trickier that it first appears. The ability to convert near-stochastic problems into stochastic ones is very practical skill for students to have.

Level 3: Examples of special states are introduced, including absorbing states and reflecting states. Long-term behavior is also analyzed. Non-stochastic effects such as situations in which there is an overall growth in the population, or growth in certain subcategories, are studied and brought within the envelop of understanding.

SUMMARY

To date, seven of the eleven projected labs have been completed, of which five have been classroom tested and revised based on student reaction. Student reaction has been positive and evaluations have been high, with many students saying the modules were the best part of the class.

There is one additional maturing component connected with the labs, a collection of perspective summaries. These summaries are relevant only after a certain subset of the labs have been completed. For example, the flow approach, the network approach, and the stochastic approach, all apply to the area of the movement of goods and services from one place to another. After completing all three labs, students are presented with a handout summarizing the three approaches, and a side-by-side comparison and contrasting of the methods. This develops the students' skill of being able to select the right approach to fit the specific problem or situation. Other such summaries relate to other subgroups of lab topics. The development of user maturity is the central goal of our laboratories.