

COLLABORATIVE PROJECTS IN CALCULUS I

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ABSTRACT:

Students often benefit from testing their understanding of calculus concepts with their classmates. This heightens interest and appreciation for the subject matter, especially among those who have had prior exposure to it in high school. In this paper short in-class projects will be shown that are designed for this purpose. Among those included will be lessons involving the relationship of functions to their derivatives, rates of change, and Riemann sums. A computer lab project using DERIVE also will allow the students to investigate numerically and graphically the effects of increasing the number of intervals in Riemann sums and of picking the test points in various ways, including randomly.

INTRODUCTION

Students come to Calculus with a variety of backgrounds, interests, and aptitudes. Some have had a full year of AP Calculus in high schools, while others have had no exposure to it at all. One way to help maintain interest for the previously exposed (and level the playing field for the rest) is to have students collaborate on joint projects. These do not necessarily have to be complex; in fact they can be quite simple and still accomplish their goal. In this paper we will give samples of three such simple projects which are calculator based and a longer one which uses the computer.

The short projects are flexible. While they are meant for use during part of a class period with discussion of results by the whole group to follow, they can also be used as follow-up homework assignments, or even, in some cases, as before class previews to set the stage for the material to be developed. Likewise, while intended for small group collaborative efforts, they can also be done on an individual basis.

The first of these is used early in the course to talk about rates of change. Besides illustrating the variety of rates the students have encountered in the past, it forces them to link rates of change with slopes of lines, first average rates with slopes of secants, and then instantaneous rates with slopes of tangent lines.

Project II can be used as indicated when talking about solutions to differential equations, or when discussing curve sketching in general. It requires students to think about the information conveyed by a graph (where y' is positive or negative), convert this to geometric knowledge (where y is increasing or decreasing), and then use this to recognize the resulting graph.

The third project is a straightforward exercise in approximating the area under a curve by Riemann sums as a prelude to the definite integral. Nevertheless, if done before any lecture on the topic, it provides a nice opportunity for collaborative work and thoughtful discussion within the groups.

The DERIVE project, on the other hand, is longer and more in-depth, requiring either a laboratory period or access to a computer laboratory after class. It is intended to have the students experiment with Riemann sums of **various** types (left, right, midpoint, and even random) to appreciate that all methods lead to the same limit. The plotted results for $n = 100$ graphically illustrate the way the Riemann rectangles closely approximate the area under the curve. The program is designed so that it can be readily modified to use whatever function is desired over whatever interval is chosen, providing the opportunity for further explorative use.

Project I

RATES OF CHANGE

Much of the differential calculus is motivated by ideas involving rates of change. When we talk about an average rate of change, we are expressing the amount one quantity changes over an interval for each single unit change in another quantity. For example, if a car travels 90 miles in two hours, it would be averaging 45 miles per hour, indicating that we expect the distance it has traveled to change by 45 miles for every one hour the time changes.

1. Express each of the following as an average rate of change:
 - a. The airfare from Wilkes Barre to San Francisco rose \$120 in the last three months.
 - b. In five trading days the stock price dropped three dollars.
 - c. At Penn State, enrollment has increased 8% in the last four years.
 - d. John expects to lose nine pounds in six weeks with his new exercise program and diet.

Rates of change and graphs:

2. Suppose the following table represents temperature values on a typical June day in Scranton. Plot the data on the axes indicated, using a scale from 7 to 18 on the horizontal axis and from 40 to 90 on the vertical axis (you can let the origin represent the point (7, 40)). Express the times on a 24 hour basis (1 p.m. is 13, 2 p.m. is 14, etc.). Use these points to sketch the graph of the temperature as a function of time on this interval.

Time	Temperature
7 a.m.	49
8 a.m.	58
9 a.m.	66
10 a.m.	72
11 a.m.	76
12 noon	79
1 p.m.	80
2 p.m.	80
3 p.m.	78
4 p.m.	74
5 p.m.	69
6 p.m.	62



3. Have **each** member of your group calculate the average rate of change of the temperature over a **different** time period (for example, one might do the period from 9 a.m to 1 p.m.).
4. Now draw the line segment between the corresponding points on your graph and find its slope. What do you observe?

Just as the slope of the segment between two points gave the average rate of change of the temperature over the interval, so too the slope of the tangent line to the curve at a point will tell us the rate at which the temperature is changing at that particular moment, the **instantaneous rate of change**.

5. Have each person in your group draw a line that seems to be tangent to your graph at the point corresponding to 10 a.m. Pick another point on this tangent line, estimate its coordinates, and use these to find the slope of the tangent line. How fast does this indicate the temperature is changing at 10 a.m.?

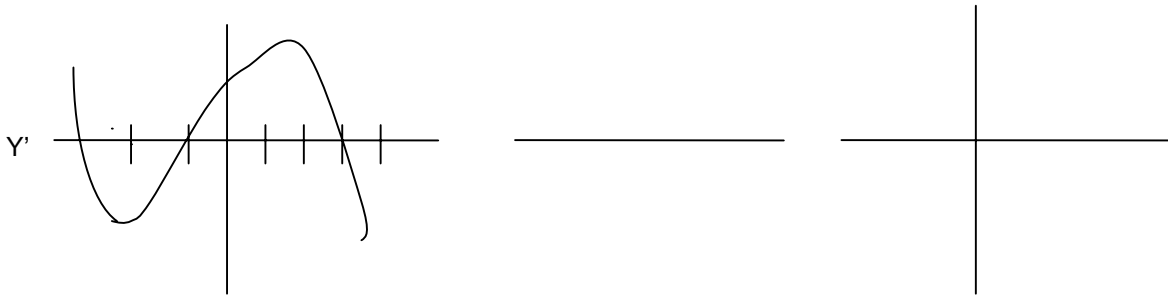
6. Compare with one another the results each of you found in the last question. Why would we not expect them all to be the same? Are they significantly different? Is this what you would anticipate?

Project II

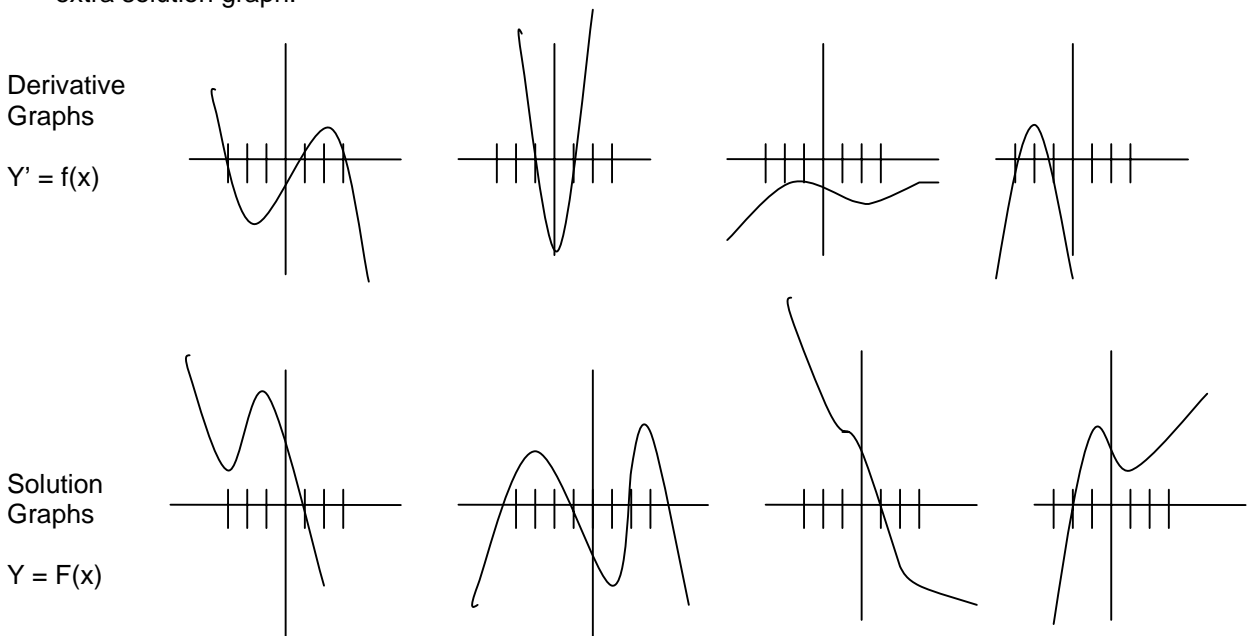
SKETCHING SOLUTION CURVES TO DIFFERENTIAL EQUATIONS

Differential equations give you information about derivatives (for example, $y' = 3x^2 + 4$). Solving them involves finding solution curves $y = F(x)$ that satisfy the differential equations. If the information in the differential equation comes in graphical form rather than as a formula, we can use our graphical analysis to sketch a rough drawing of what y might look like (**a solution curve**).

- Suppose y' has the graph below. **Working together**, do a sign chart for it on the real number line indicating where it is positive or negative and then sketch three graphs of y that would be consistent with this. Where are the local maxima and minima occurring?



- Divide up** the derivative graphs below among your group and match them to the appropriate solution graphs they could come from. **Check** that your answers are consistent with one another. If there is a derivative graph left over analyze it together to make sure it leads to the extra solution graph.



- Notice that the solution curves are concave upward on regions where the graph of y' is increasing and concave downward on regions where y' is decreasing. Why is this so?

Project III

APPROXIMATING AREAS WITH RECTANGLES

Draw the area under $y = 5 - x^2$ between $x = 0$ and $x = 2$.

Trying to find how much area is involved:

1. Divide the interval from 0 to 2 into four equal pieces.
 - a. What are the endpoints of the subintervals?
 - b. Build rectangles using right hand endpoints of the subintervals - go from x-axis up to curve to get height.
 - c. What is the sum of these areas?
 - d. How does this compare to the area under the curve?
 - e. Build the rectangles using the left hand endpoints.
 - f. How does this sum compare to the area under the curve?
 - g. Can you give values that the area under the curve will lie between?
 - h. What will happen if you increase the number of subintervals into which $[0, 2]$ has been split (for example, if you used eight instead of four)? Does it seem as though the estimates will improve or get worse, or is there no apparent connection?
 - i. Is there anything about the function $y = 5 - x^2$ that makes the estimates using the left hand endpoints larger than those using the right, or would this be the case with any function?

Project IV

USING RIEMANN SUMS TO APPROXIMATE AREA (DERIVE VERSION)

In this project you will be using the DERIVE software to experiment with approximating the area under a curve using Riemann sums (in this instance, the sums of areas of rectangles).

After you have started DERIVE for Windows, put the disc containing the program RIEMBASC in Drive A and open the program. It is set to work on approximating the area under $y = x^2 + 1$ between $x = 0$ and $x = 2$. Line 6 and lines 2 and 3 specify this function, but you could highlight these and edit the expressions to pick a new function or interval.

For your information, here is how some of the lines in the program relate to our usual ideas:

Line 4 uses d to represent what we would normally call Δx . Since line 12 has n set to 4, we are initially dividing the interval from 0 to 2 into 4 equal pieces, each $\frac{1}{2}$ a unit long. As you work on the project, you will be using different values for n .

The a_i defined in line 5 is what we would normally think of as the left-hand endpoint of interval i . For example, when $i = 1$, it gives the left hand end point a ; when $i = 2$, it would add Δx to a and give the end of the first interval or the beginning of the second, that is to say, the left-hand endpoint of the second interval.

$\text{RIEMANN_RECT}(r, d, t)$ in line 13 gives the four corners of a rectangle d units wide and t units high if the base is on the x -axis with the lower left-hand corner at the point $(r, 0)$.

LEFT HAND SUMS

1. Open the Plot window by hitting the 2-D Plot key (the button that looks like the graph of a curve on the x - and y -axes), or choosing Window on the taskbar and then picking New 2-d Window.
2. You will probably have to change your range of values to be able to go up to the $y = 5$ value you get for $y = x^2 + 1$ when $x = 2$. To do this, choose Set from the taskbar, and then Range. You might set the Top to 6.5, Left to -1.5 , Bottom to -1.5 , and Right to 3.5, TABbing from one setting to the next; then Set the Scale to .5 for Horizontal and 1 for Vertical. Pick Window and Tile Vertically to get the Algebra and Plot screens side-by-side.
3. Put your cursor in the Algebra window and click it to activate this side. Now Author #8 to copy over and activate this definition of t_i , the point in the interval where the function is to be evaluated (this is what is sometimes labeled as " x sub i star"). QUESTION: Explain why this definition of t_i as a_i gives the left-hand point of the interval for each i .
4. Highlight line 14 and hit the = button to simplify it, or choose Simplify from the taskbar at the top, pick Expression, and type in #14.
5. To plot the rectangles involved in this left-hand approximation, click on the Plot window and then on either the Plot button (graph on the x - and y -axes), or choose Plot! from the taskbar. QUESTION: Is this approximation going to overestimate or underestimate the area under $y = x^2 + 1$? Does it seem like a good estimate or not?
6. Go back to the Algebra window, Author r_{sum} , and Simplify it. QUESTION: Looking at lines 11, 9, and 4, explain why r_{sum} gives the area within the rectangles you just plotted.

7. Now Author $n:=8$. Repeat your actions in 4 and 5 above. This will superimpose the new set of rectangles on top of the old. Print this graph. Find the new rsum as you did above. QUESTION: How does this compare to the earlier estimate; is it a better or poorer approximation? Shade the portions of the rectangles in your graph that were not included in the first approximation. What is the area of this shaded portion?
8. Now Author $n:=100$ and repeat what you did in 4, 5, and 6. QUESTION: What do you observe?

RIGHT HAND SUM

1. To set up the right hand sum, redefine $t1$ using Author #16.
2. Author $n:=4$, Simplify line 14 and Plot the result; go back to the Algebra window and find rsum. QUESTION: Is this approximation going to overestimate or underestimate the area under $y = x^2 + 1$?
3. Author $n:=8$. Repeat your actions right above. Print the graph. QUESTION: How do these rectangles compare to the ones at the last step. Is this rsum a better or poorer approximation to the area under the curve? Shade the portions of the rectangles in your graph that were included in the first approximation, but not in the second. What is the area of this shaded portion?
4. Now Author $n:=100$ and repeat the work you did in 2 above. QUESTION: What does the graph suggest about the rsum you found?

OTHER SUMS

1. Use Author #18 to change $t1$ to pick the midpoint of each interval for evaluation of the function and find rsum (n is still 100); use the approximate button (wavy equal sign) to get a decimal approximation to this.
2. Now use Author #20 to have $t1$ pick the evaluation point in each interval in a random manner. Again approximate rsum. Approximate rsum a second time. QUESTION: Why are the two rsums different?
3. List the five values you have gotten for rsum with n as 100 (left, right, midpoint, and the two random sums). QUESTION: What do you guess the area under $y = x^2 + 1$ from 0 to 2 to be?
4. QUESTION: Summarize what all this work shows about Riemann sums.

DERIVE PROGRAM TO GRAPH RIEMANN RECTANGLES

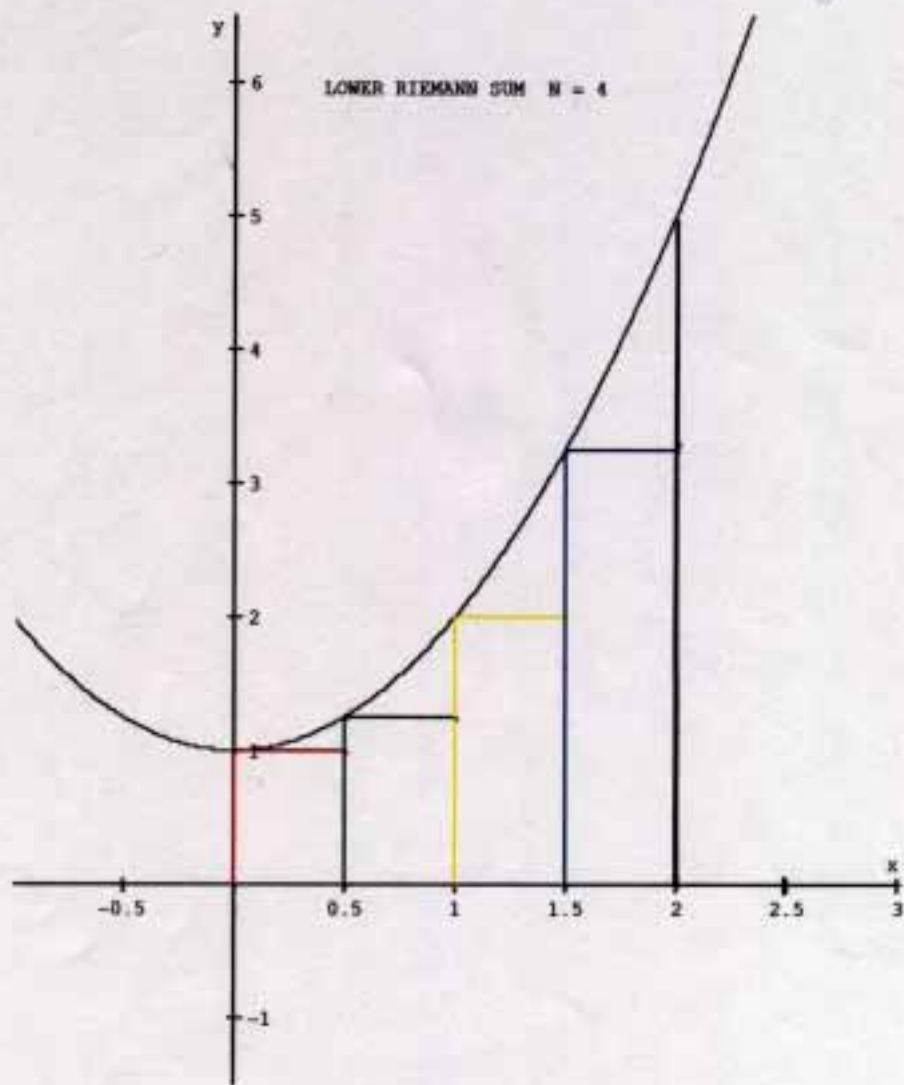
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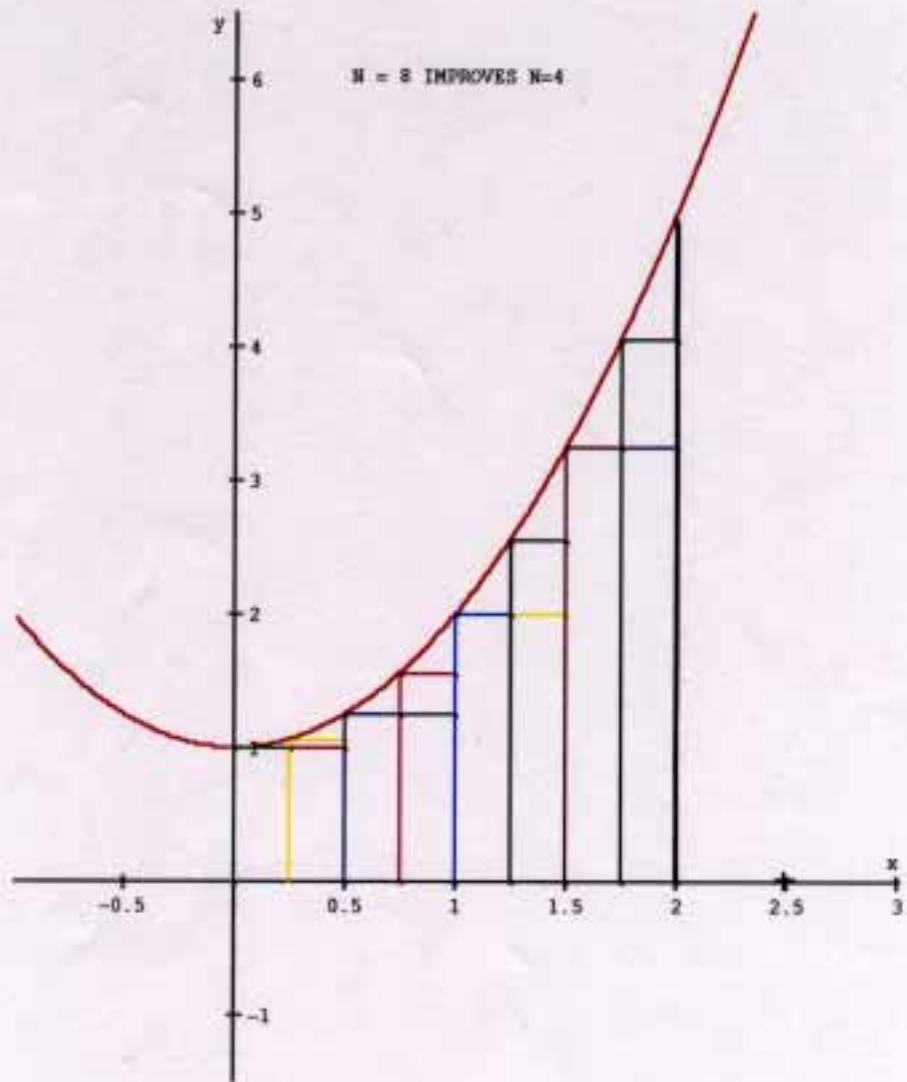
#1: precision := APPROX(IM(a)) · t · e
#2: a:= 0
#3: b:= 2
#4: d :=  $\frac{b-a}{n}$ 
#5: a1 := a + d · (i - 1)
#6: F(x) := x2 + 1
#7: "left sum"
#8: t1 := a1
#9: h := F(t1)
#10:  $\sum_{i=1}^n h \cdot d$ 
#11: rsum :=  $\sum_{i=1}^n h \cdot d$ 
#12: n := 4

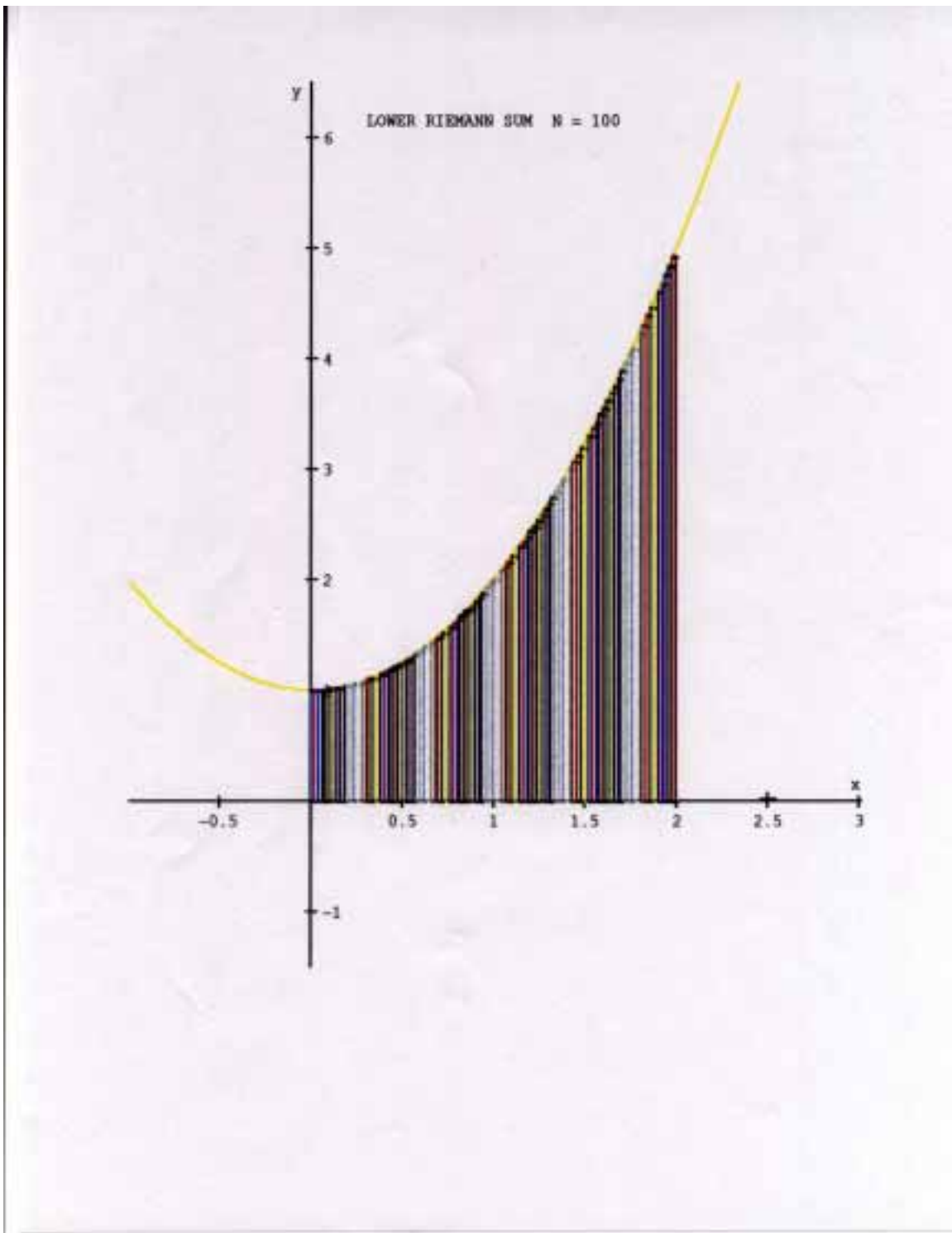
#13: RIEMANN_RECT(r, d, t) :=  $\begin{bmatrix} r & 0 \\ r & t \\ r+d & t \\ r+d & 0 \end{bmatrix}$ 

#14: ares_approx := [F(x), VECTOR (RIEMANN_RECT(a1, d, h), i,1,n)]
#15: "right sum - use Author #16"
#16: t1 := a1 + d
#17: "midpoint sum - use Author #18"
#18: t1 := a1 +  $\frac{d}{2}$ 
#19: "arbitrary sum - use Author #20"
#20: t1 := a1 + d · RANDOM(1)
#21: t1 := a1
#22: "highlight line #14 and Simplify"
#23:  $\left[ x^2 + 1, \left[ \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ \frac{1}{4} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{3}{4} \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 2 & 2 \\ 2 & 0 \end{bmatrix}, \left[ \begin{bmatrix} 0 & 0 \\ 0 & \frac{13}{4} \\ \frac{13}{4} & 0 \end{bmatrix} \right] \right]$ 

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