

VISUAL LINEAR ALGEBRA: WEB-BASED EXPLORATION APPLETS

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ABSTRACT: Interactive applets for linear algebra have been developed at the University of Michigan–Dearborn to help students not only develop basic skills (for example, learning to produce matrices which manipulate other matrices), but also explore more advanced topics (discovering eigenvectors using visual feedback, and viewing 3-d objects through their Singular Value spectral series expansion.)

INTRODUCTION

A linear algebra project at the University of Michigan – Dearborn has developed and is developing pedagogical materials for linear algebra. Our University is a mid-sized (6,600 undergraduates) state-supported, commuter university located in metropolitan Detroit. The commuter aspect provides particular incentive for the development of web-based course support materials, suitable for students to use on their home computers away from the campus. A materials development project funded by a grant from the National Science Foundation is currently underway, and has as one of its primary components, the development of a web-based presentation of support materials.

GENERAL DESCRIPTION OF THE PROJECT

The project began when engineers and computer professionals for whom we were consulting, and with whom we spoke at SIAM meetings, asked us why the linear algebra they learned in college was so different from the linear algebra they use in their work. Most courses currently provide the student with a set of matrix manipulation skills, combined with a body of theoretical knowledge. Very little time is given to teaching students to think in a linear algebra mode. Our project implements changes in the current course, motivated by the goal of teaching students to operate at a higher algebraic and geometric level, both in the area of theoretical understanding and in the area of problem solving. This is a sophisticated goal, not solved by simply including more applications.

The main innovations of the overall project include:

- A de-emphasis of component-wise manipulation of matrices, and instead an emphasis on block-matrix manipulation. One consequence of this approach is a prominence in the role of matrix factorizations.
- A strong linkage between geometry and linear algebra.
- Methods of proof and numeric techniques are constructive at all times, and are consistent with computational algorithms that are used in practice.
- Applications, in addition to illustrating various aspects of matrix and linear algebra, emphasize the process of mathematical modeling. The modeling emphasis is of two types: some models are showcased because of their ability to address general classes of seemingly different situations, while other models are presented in stages to illustrate how to accommodate successively more realistic information.
- Computer technology is called upon to promote a macro-level viewpoint by eliminating the tedium of matrix manipulation.

The innovations listed above have a major common thread, which is the visual approach to linear algebra, and so it becomes so it become quite natural to augment our materials with the use of computer displays to illustrate the concepts. To reach the goal mentioned earlier of teaching students to operate at a higher algebraic and geometric level, we are developing not only written materials, but also an important component of web-based course support materials. The topic of this paper is a description of the web-based aspects of the project.

THE WEB-BASED EXPLORATIONS

Interactive applets developed at the University of Michigan – Dearborn help students not only to develop and practice their basic skills, but also to explore more advanced topics. In this paper we shall discuss six such applets, and then provide information on how to gain access to viewing other applets, which we have developed. Three of the six applets discussed below are samples from the basic skills level, and the other three are samples of our exploratory applets at a more advanced level.

BASIC SKILLS LEVEL

Applet #1) Learning and practicing binary matrix operations:

The process of learning an operation such as matrix multiplication is made easier and faster when the student is provided with a way to practice his or her developing skills, all the while being given immediate feedback on his or her mistakes, and instructions (when requested) on how to proceed, all provided by an infinitely patient computer. This is of course a standard applet at many linear algebra sites. Our version is as follows:

The student selects a binary operation (+, -, *) and then picks the desired size of the matrices to be used. Matrix entries can be computer generated at any of three levels (easy, medium, or hard), or can be entered by the student. The two input matrices are displayed on the screen with the operation sign between them. If the sizes for the input matrices are not compatible with the given operation, the student is informed, and asked to correct the situation. The student then types in the size of the expected result, and if that is correct, a matrix with blank entries is presented on the right side of the equal sign. The student clicks on any blank entry in this new matrix and types in his or her answer. At any point, the student can click on a special button to check the correctness of the work typed in so far. A help button is also always available to provide students with an informative help box of instructions on how to proceed. Color-coded feedback (green-ing the entries that are correct and red-ing the entries that are incorrect) provides an additional avenue for learning reinforcement, and also makes the screen appear livelier.

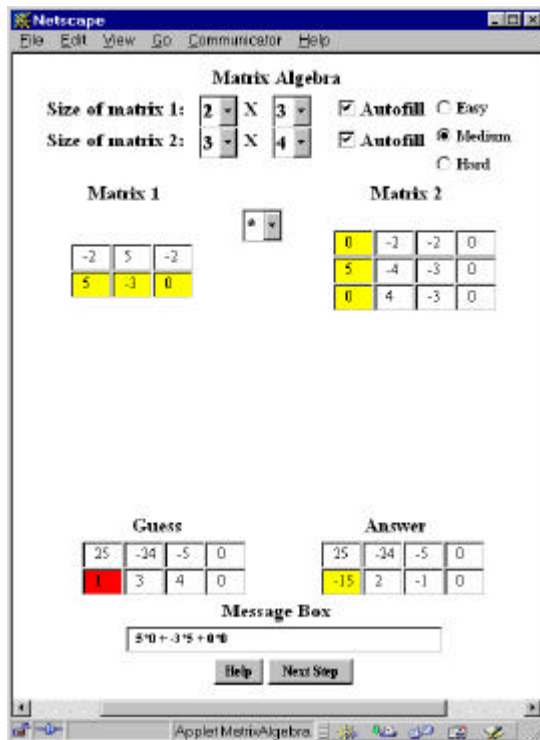


Figure 1.
Binary Matrix Operations

Applet #2) Learning and practicing unary matrix operations:

This applet is quite similar to the one above for binary operations, but deals with the unary operations (A^{-1} , A^T , $|A|$). Again, the student can select the size and entries of the original matrix, or the matrix can be autofilled. As above, the input matrix is displayed with its entries, and the goal matrix is displayed with its blank entries. Students fill in the blank entries in an interactive manner, and can check the correctness of their answers at any time in the process. One new choice for students for the A^{-1} operation is the option : “does not exist”. A help button is always available for more information. Answers are colored green for correct and red for incorrect. In both this applet and the previous one, there is always the option available to “show answer”.

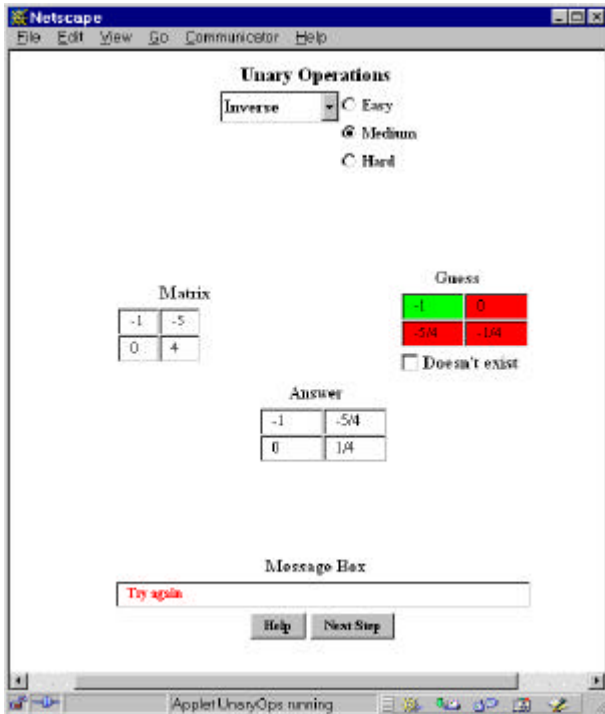


Figure 2.
Unary Operations Applet

Applet #3) Learning to produce matrices which manipulate other matrices

A goal of this project is to teach students to think in a linear algebra manner. One step in this process is for students to learn how to make a desired action occur by producing the correct matrix and operation to accomplish this action.

To begin, the student is asked to select one of the following categories: 1) Isolate a row or a column, 2) Permute two rows, or two columns, 3) Permute several rows, or several columns, 4) Scale by rows, or by columns, 5) Add one row to another, or add one column to another, or 6) Mixed Bag. Each of these categories presents an interactive arena in which to practice the selected skill. For example, if category 1) is selected (Isolate a row or a column) a 5 by 5 matrix with random entries is presented on the screen. Then one of the rows or columns from this matrix is displayed on the right side of the screen. The student is asked to find a special "selection vector" which when multiplied by the original matrix, will result in this goal row or column that is displayed. The desired "selection vector" is positioned in the correct place (to the right of the original matrix to isolate a column, and to the left to isolate a row) but to begin, all entries are set to 0. The student is invited to try to type the proper changes into the entries of the selection vector that will make the matrix multiplication result in the goal matrix. At any time the student can press a button to see what the result of the current selection vector will be. After a few tries, the student usually arrives at the correct selection vector, and thereby has mastered a useful skill in matrix manipulation. As described in the first

applet, color-coding of correct and incorrect entries in the selection vector provides an additional avenue for feedback.

Interactive exploration in categories 2) through 6) are available in a similar format. For example, in 3), a random matrix is displayed (the “original”), along with a copy of this matrix having several rows (or several columns) interchanged (the “goal”). Then a square matrix of the correct size is displayed (the “selection matrix”) in the appropriate position, either to the left or to the right of the original matrix. To begin, all entries of the selection matrix are 0 except down the main diagonal where 1’s appear. The student is to change a few entries in this selection matrix so that when the original matrix multiplies it, the result will be goal matrix. At any time the student can press a button to see what the current result would be.

The payoff in these explorations is that the student gains control. There is a major difference between being told: “this matrix multiplied by another one does thus and such” and being told to play around until you have gained the skill of making thus and such happen. The first is passive learning and the second is active learning. Whenever possible, active learning is preferred, and certainly in categories 1) through 6), active learning is possible and preferred. This approach of thinking of a matrix as a verb which accomplishes a certain action, is a way of empowering students to think in a linear algebra manner.

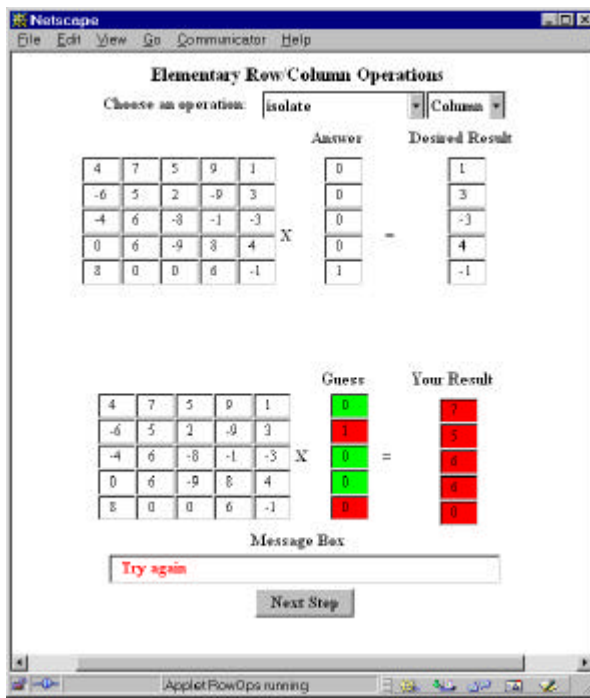


Figure 3.
Elementary Row/Column Operations

Applet #4) Finding approximations to eigenvectors and eigenvalues using visual feedback

The viewing of a matrix as an action, which transforms vectors, or matrices into other vectors or matrices, represents a maturity level worth developing in linear algebra students. To help develop this outlook, the computer can be used to display the vector x and the vector Ax . By comparing these two vectors, the student gains information about the nature of A .

The eigenvector / eigenvalue applet begins with a display of a x - y coordinate system with limits $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$, on which is displayed a faint green unit circle. When the student presses the NEXT button, a 2 by 2 matrix is selected from within a family of matrices having real eigenvalues in the range of 0 to 5. The student then clicks on any point on the unit circle, and as a result, a vector x is drawn in red starting at the origin and ending at the point clicked, and also the vector Ax is drawn in blue, also starting at the origin. In this way, the vector x and its image Ax can be compared. The student is encouraged to try clicking several different points on the unit circle, and each time a click is made, the previous vector and its image disappear to make room for the new red x and its blue image Ax . In this way, the student can get a feel for how the matrix A manipulates vectors which are going in different directions.

Then the student is asked to try to find (by clicking on the points on the unit circle) the x for which the vector Ax (blue) points in the same direction as the vector x (red). For such an x , Ax is just some multiple of x , that is $Ax = cx$ for some real constant c . This special vector x which the student has discovered, is the eigenvector. Throughout the exploration, the coordinates of the current head of the vector x , and also of the current head of Ax are being displayed in data windows off to the side. Also the length of Ax is displayed. So when the students discover this special x (the eigenvector), they also have found the corresponding eigenvalue (up to sign), since $|Ax| = |c||x| = |c|$, since $|x|=1$. When the students have found what they believe are the eigenvectors and eigenvalues, they are asked to type them in, and are then informed whether or not they have succeeded (to within a tolerance of 8%.)

This exploration has a further option, which allows the student to input the entries for A themselves instead of having the computer generate the entries. This allows an exploration of the geometry similar to that described above, but for the cases of repeated eigenvalues, or negative eigenvalues, or complex eigenvalues.

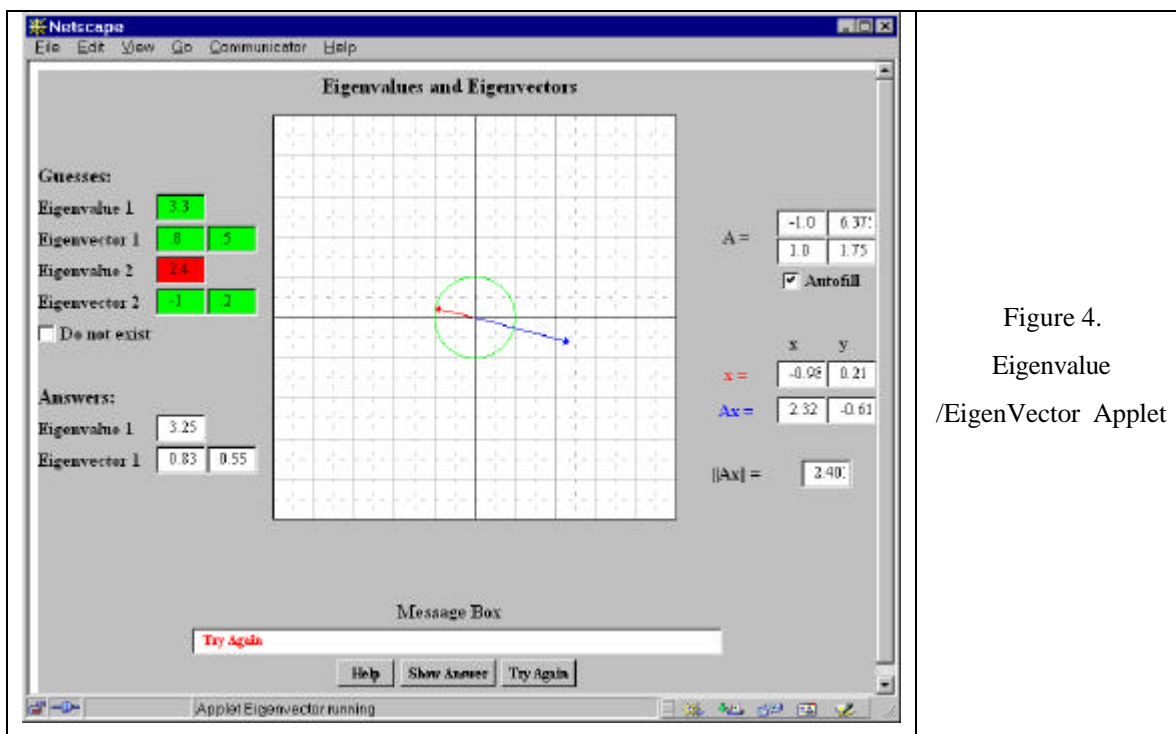


Figure 4.
Eigenvalue
/Eigenvector Applet

Applet #5) Representation of 3-D objects using the Singular Value spectral series.

The role of the Singular Value Decomposition has increased significantly in importance during the past twenty-five years [1, 2, 3], and now occupies a central place in matrix algebra. It joins the LU decomposition (the optimal way to solve $\mathbf{Ax}=\mathbf{b}$ for a square or underdetermined system), the QR decomposition (a numerically stable way to solve $\mathbf{Ax}=\mathbf{b}$ for an overdetermined system), and the study of eigenvalues and eigenvectors (the way to diagonalize a matrix and thereby decouple a system of equations) as areas with which students must be comfortable if they are to operate in linear algebra at a useful algebraic and geometric level.

One application of the theory of singular values occurs when information is stored in matrix format, and what is desired is the best way to sort out the important information from the unimportant information stored in the matrix. The Singular Value Spectral Theorem states that

$$\mathbf{A} = s_1 \mathbf{u}_1 \mathbf{v}_1^T + s_2 \mathbf{u}_2 \mathbf{v}_2^T + \dots$$

where $s_1 \geq s_2 \geq s_3 \geq \dots \geq 0$ are the singular values and the \mathbf{u}_i 's and \mathbf{v}_i 's are the left and right singular vectors, respectively. Since the s_i 's are decreasing in size, the first term in the sum above is generally the most important, the second is the next most important, and so forth. The effects of reconstructing \mathbf{A} from its spectral sum have thus been sorted in order of decreasing importance.

The theory behind the singular value decomposition involves many details which, although important, can be difficult for students, and can obscure from them the underlying value. The learning of the

theory is not alone sufficient to make students take ownership of this area. Our materials include the following applet to help students to visualize the ideas discussed above.

The applet begins with an object, the physical description of which has stored in matrix format. More precisely, the data is the set of vertical offsets taken above each lattice point of a rectangular grid. The data sets we used came from a variety of sources. A coordinate sampling machine in the engineering department of our University produced some data sets locally. Another data set is a model of a vortex, which appears in the study of interlaminar flow. Yet another set was produced using AutoCAD. Students can step through a sequence of singular value approximations to the original data set. The students click on a button to view the next or previous data set and for each choice they view a slightly different approximation to the object as well as a subset of the data used to generate the graph. For instance, if they click to the 2^{d} approximation, then the surface is displayed with vertical offsets given by the two-term approximation for $\mathbf{A} \sim s_1 \mathbf{u}_1 \mathbf{v}_1^T + s_2 \mathbf{u}_2 \mathbf{v}_2^T$. Students can experiment, picking fewer or more terms in this series, and observe the corresponding convergence in the 3-D depiction of the surface. In the words of Gilbert Strang [4] describing this process: “The pictures are really striking, as more and more singular values are included. At first you see nothing, and suddenly you recognize everything.”

As an additional component, students can choose to view the surface corresponding to any individual term in the spectral series instead of the sum of all the terms 1 through n. In this way they can compare the relative contribution of the various terms in the sum.

This applet is in some sense not as interactive as the others. For each object, the singular values and singular vectors have been stored – they are not computed on the spot, and the three dimensional pictures, one corresponding to each new term in the spectral theorem, are likewise stored, and not computed on the fly. So this applet is really just a display of several different options in a previously worked out problem. Nevertheless, the student will seem to be in control, and in some limited sense is in control. What is certain about the applet is the pedagogical value. The point is clearly and visually made of the significance of singular values in a way the student will not soon forget.

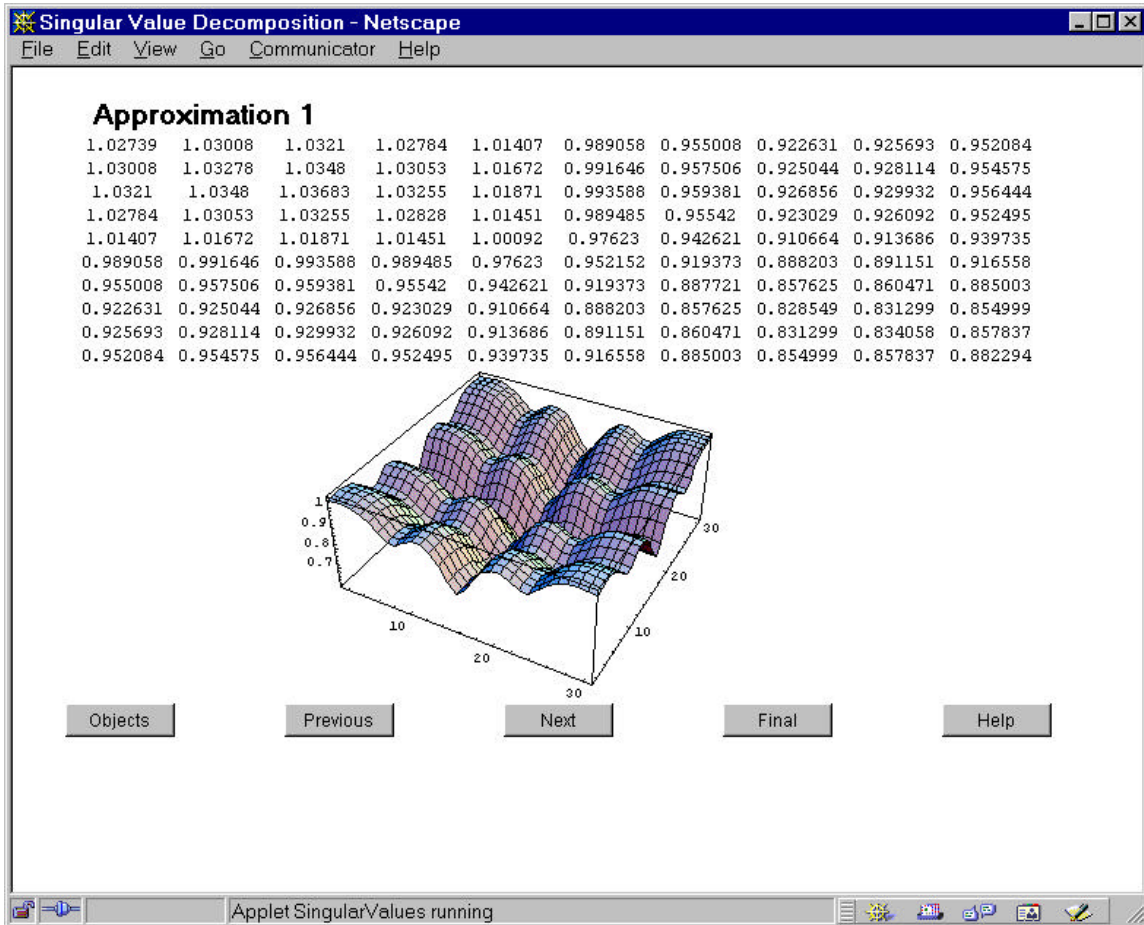


Figure 5.

Viewing Singular Values

Applet #6) The geometry of Singular Values and Singular Vectors.

This applet concentrates on illustrating some of the geometric properties of singular vectors. Given a 2 by 2 matrix A , there are two left singular vectors u_1, u_2 , and two right singular vectors v_1, v_2 . The left singular vectors are the major and minor axes of the ellipse formed by the image of the unit circle under the transformation AA^T .

The applet displays an x-y coordinate system on which appears a faint green unit circle. A 2 by 2 matrix is selected and displayed. Students click on a point on the green unit circle, and in response the applet displays a blue dot at the position of the image of this point under the transformation AA^T . After clicking several points, an ellipse becomes evident. The student then explores this ellipse (again by clicking points) to locate as nearly as possible its major and minor axes. These two axes are the two left singular vectors of A . As described by the information in the HELP box, once a singular vector has been located, for instance the major axis of the ellipse, the corresponding singular value is the square root of the length of that major axis. And since the applet not only displays the

coordinates of the most recent blue dot, but also has a box recording the numeric value of the distance of that blue dot from the origin, it is easy for the student to use this tool to find the length of the major axis, and then take the square root to ascertain the singular value. Students are asked to type in the values of the singular vectors and the singular values, and are then informed whether or not they have succeeded (to within 8% tolerance).

Currently under development is a similar applet for 3 by 3 matrices A , in which the green unit circle for the 2 by 2 case above has been replaced by a green unit sphere, and by clicking on points on this sphere, their images under the transformation A^T appear as blue points on an ellipsoid. The three singular vectors for A are the major and two minor axes of this ellipsoid, which students can discover through exploration.

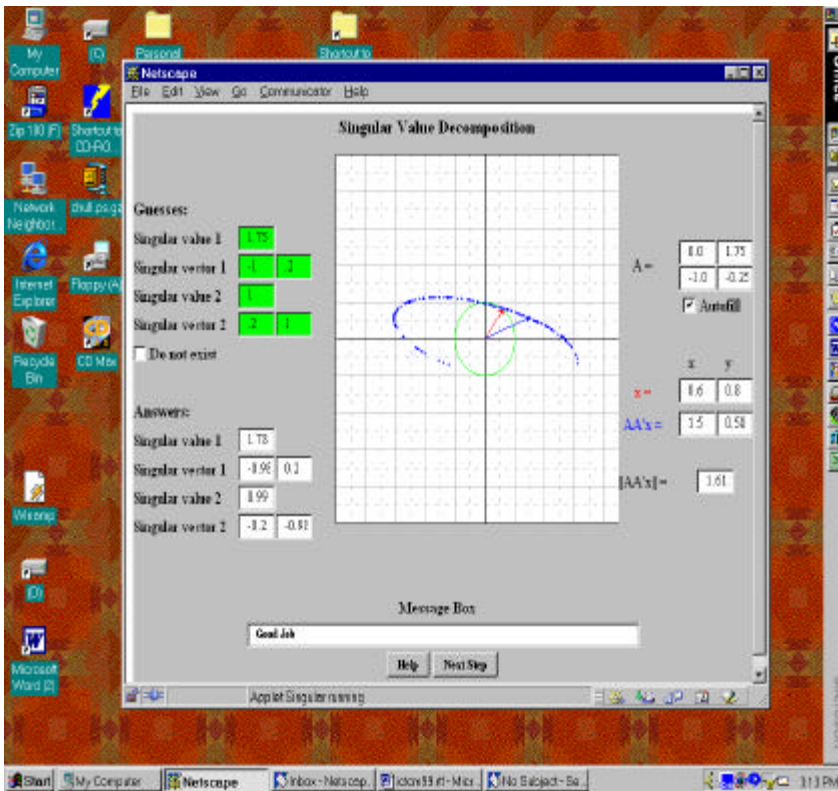


Figure 6.
Singular Value/
Singular Vector
Geometry

POSTING OF LECTURES ON THE WEB

One of the authors, Dr. Elenbogen, currently posts his computer science lectures on the web, including spontaneous remarks typed in during the on-going lecture. Students can then access that day's lecture from home to correct or replace their own necessarily more abbreviated notes. Although the other two authors have not yet adopted this approach, information on ways to implement a successful approach to this use of the web can be found by inquiring at boss@umich.edu

CONCLUSION

Our project was designed to help students learn to think in a linear algebra manner. The providing of a visual context for an active learning environment through the web suits this purpose very well. The applets described above are helping our students achieve the goal of operating at a higher algebraic and geometric level in dealing with problems of linear algebra.

An entire course based on the visualization of matrix operations, factorizations, and other matrix algebra concepts, is currently being taught at the University of Michigan – Dearborn. It interweaves the approaches described above with the other innovations mentioned at the beginning of this paper. A textbook, “Visual Linear Algebra” based on this approach will be available in Fall 2001. A preliminary version of these and other applets can be found www.engin.umd.umich.edu/~boss/linalg/linalg.shtml.

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