# PRODUCT OF SHEARS

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In our college geometry course, shears are a small component of our treatment of transformational geometry. Yet, shears are important because they counteract the commonly held belief that all transformations are isometries – or at least similitudes. While I was consulting for the CAS-Intensive Mathematics Project[1] this summer, a colleague posed a question: "Are you aware that rotations are implemented on some dynamic geometry tools as a product of three shears?" After my initial response of skepticism, I began to explore the question.

## A Theorem

My investigation resulted in the following theorem: *A rotation about the origin is the product of three shears*. In the proof I used shears about the *x*-axis and *y*-axis. The rotation as a product of shears is $\rho_{O,\theta} = S_1 \circ S_2 \circ S_1$ with equations

$$\begin{cases} x' = \left(\dfrac{1-b^2}{1+b^2}\right)x + \dfrac{2b}{1+b^2}y \\ y' = \dfrac{-2b}{1+b^2}x + \left(\dfrac{1-b^2}{1+b^2}\right)y \end{cases}$$

where $S_1$ is a shear about the *x*-axis with factor *b* and $S_2$ is a shear about the *y*-axis with factor $a = \dfrac{-2b}{1+b^2}$. Figure 1 gives an example of this product; that is, C'''D'''E''' is a rotation of CDE about the origin through directed angle $\cos^{-1}\left(\dfrac{1-b^2}{1+b^2}\right)$.

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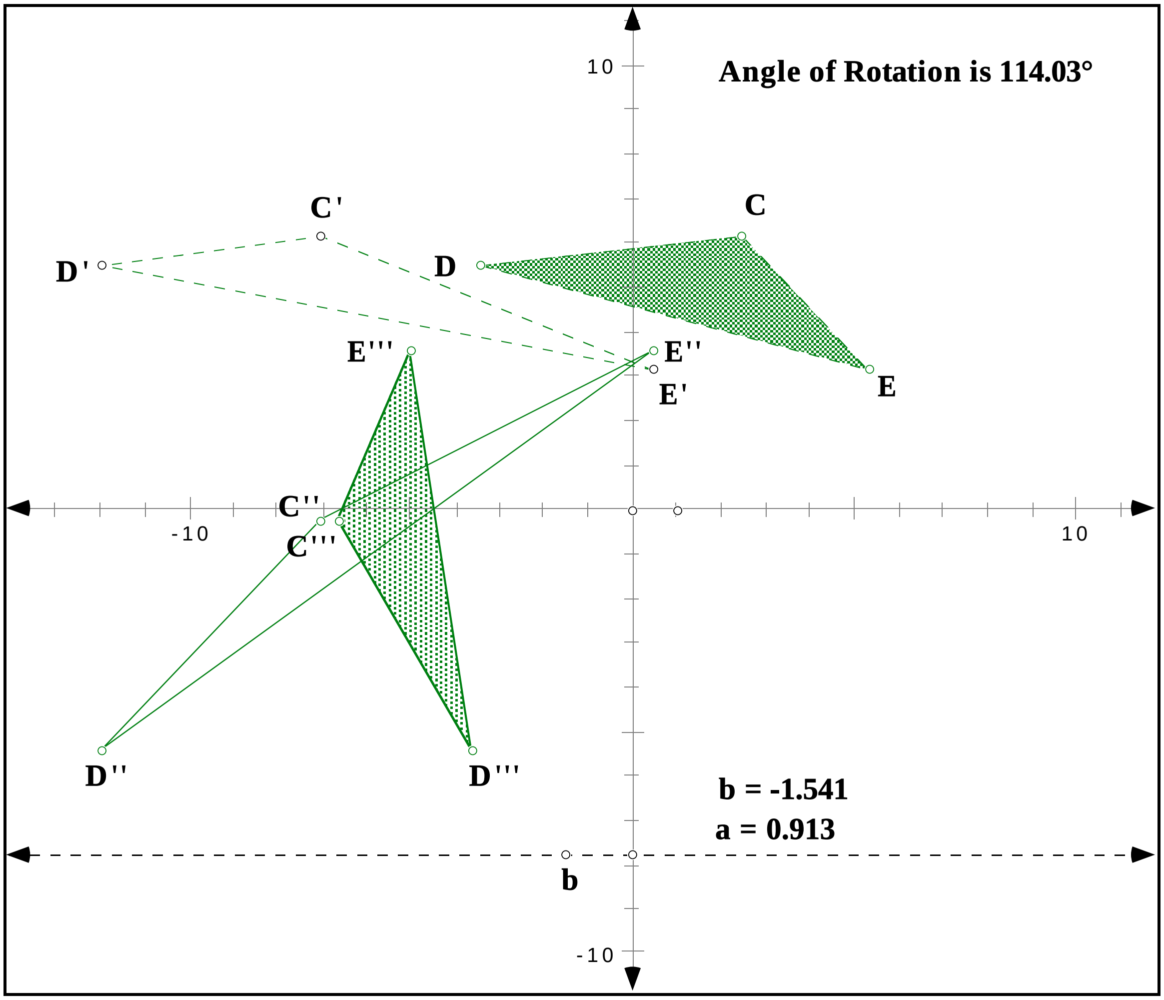


Figure 1. CDE transformed by a product of three shears with factors *b* and *a*,

$$\rho_{O,\theta}(\mathrm{CDE}) = \left(\mathrm{S}_1 \circ \mathrm{S}_2 \circ \mathrm{S}_1\right)(\mathrm{CDE}) = \mathrm{C}'''\mathrm{D}'''\mathrm{E}'''.$$

## Dynamic Geometry Simulation

Simulating the relationships in this theorem with a dynamic geometry tool like *The Geometer's Sketchpad*™ (Jackiw, 1995) can be a Motivation Provider (Zbiek, 1998) as instructor and students explore the meaning of the mathematics underlying the theorem. It is also a stimulus while thinking about transformational geometry (Zbiek, 1999). Initially, our discussion focuses around the observation that the product of these three shears, which individually are not isometries, results in a rotation which is an isometry. This can lead to further investigations surrounding the question: Are there other compositions of functions which have properties that the individual (component) functions do not have?

Other lines of reasoning can also arise. Domain and range issues emerge while we use the simulation and examine the angle of rotation as the shear factor *b* gets very large or very small. As we drag the slider (point *b*) we never quite seem to get to a rotation of ±180°. Taking a closer look at the quotient, $\frac{1-b^2}{1+b^2}$, results in the realization that the limit as *b* approaches infinity of this rational expression is -1 and $\cos^{-1}\left(\frac{1-b^2}{1+b^2}\right)$ will never be 180°. Likewise, $\lim_{b\to\infty}\frac{2b}{1+b^2}=0$ and $\sin^{-1}\left(\frac{2b}{1+b^2}\right)$ will never be 180°. We return to shears $S_1$ and $S_2$ having factors $b=-\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$ and $a=\sin(\theta)$, respectively, where $0\theta<180°$. Figure 2 displays a rotation implemented as a product of three shears whose factors are a function of input measure θ (slider point θ).

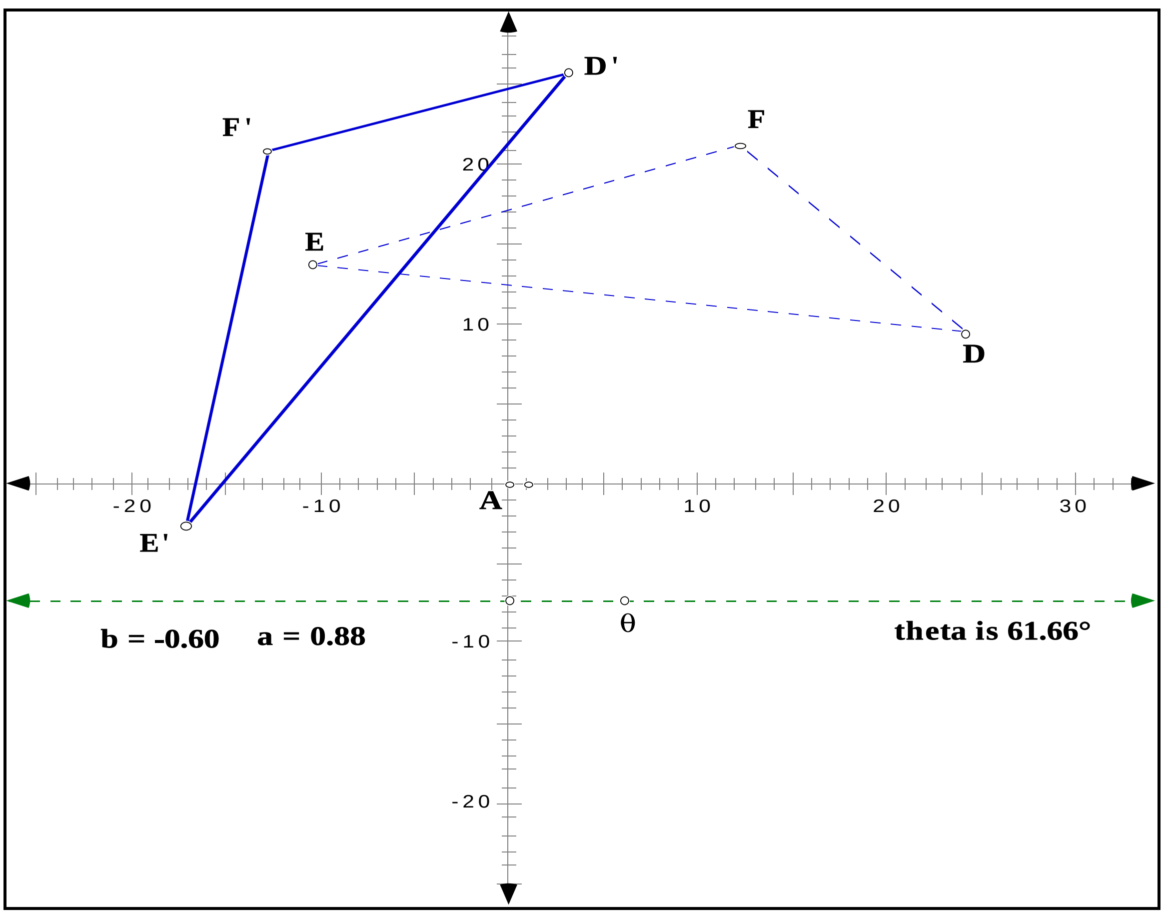


Figure 2. DEF transformed by a product of three shears with factors *b* and *a* calculated from input value θ (slider point θ); $\rho_{O,\theta}(DEF) = D''E''F''$.

## Conclusion

This technology-based approach to shears addresses several teaching issues. First, it provides a natural forum for the importance of trigonometric functions and their inverses. The use of technology also raises questions with respect to domain and range. This happens, for example, when we consider possible values of the shear factor, *b*. Technology similarly raises the question of when a composition of functions can have properties that the individual (component) functions do not have. For example, a rotation is an isometry but a shear is not. Lastly, and perhaps most importantly, there are natural ways to get students and prospective teachers into using technology as tools for teaching and into understanding the mathematics that underlies the technology these future teachers and their students will use in mathematics classrooms.

## References

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Zbiek, R. M. (1998). Do they really use graphics/symbolic calculators, or just hold them? In G. Goodell (Ed.), *Proceedings of the ninth annual international conference on technology in collegiate mathematics*, (pp. 538-542). Reading, MA: Addison-Wesley Publishing.

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