CORDIC: Elementary Function Computation Using Recursive Sequences

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Many of us who teach calculus and mathematical topics that use calculus have taken for granted that hand-held calculators use Taylor series or a variant to compute transcendental functions. Thus, it was a surprise to learn that this was not the case. The CORDIC method (Coordinate Rotation Digital Computer) was developed by Jack Volder [6] in the late 1950's. Hewlett-Packard was quick to realize the usefulness of this method; it required only the most efficient processes to compute values of the standard transcendental functions.

It should be noted at the outset that, while this presentation presumes base two arithmetic, calculators use base ten arithmetic with specially designed chips that use binary coded decimal (BCD) arithmetic. This was done to reduce the need for limited storage in the early years. While storage is no longer a problem, the algorithms are very efficient and adequate for calculator use.

Many of the papers on CORDIC that I have located were written for an engineering audience. These include the original paper by Volder and, subsequently, papers by Linhardt and Miller [1], Walther [7], and Schmid and Bogacki [4]. Two sources of information on CORDIC for a mathematics audience are articles by Schelin [3] and the COMAP article by Pulskamp and Delaney [2].

What is CORDIC?

Define a sequence of triplets $\{ (x_k,y_k,z_k) \}$ recursively for $k \geq 0$ by

$$\begin{cases} x_{k+1} = x_k - m\delta_k y_k 2^{-k}, \\ y_{k+1} = y_k + \delta_k x_k 2^{-k}, \\ z_{k+1} = z_k - \delta_k \varepsilon_k . \end{cases} \tag{1}$$

The ε_k, the initial point (x_0,y_0,z_o), and m determine the function and the point where that function is to be computed. The δ_k ($= \pm 1$) are chosen during iteration so that we always approach the desired value. In particular, m = 0, +1, or -1 with m = 0 to obtain a product or quotient, m = 1 to obtain $\sin(\theta)$, $\cos(\theta)$, or $\tan^{-1}(u)$, and m = -1 to obtain $\sinh(u)$, $\cosh(u)$, e^u, $\tanh^{-1}(u)$, $\sqrt{u}$, and $\ln(u)$; I have used u here so as to avoid confusion with the variables in the recursion process. The specifics are shown in table 1.

	Rotation ($z_k \to 0$) $\delta_k = \begin{cases} 1 & z_k \geq 0 \\ -1 & z_k < 0 \end{cases}$	Vectoring ($y_k \to 0$) $\delta_k = \begin{cases} 1 & y_k < 0 \\ -1 & y_k \geq 0 \end{cases}$
$m = 0$ $\varepsilon_k = 2^{-k}$ $k \geq 0$	x_0, z_0 given, $y_0 = 0$ implies $y_{n+1} \approx x_0 z_0$	x_0, y_0 given, $z_0 = 0$ implies $z_{n+1} \approx y_0/x_0$
$m = 1$ $\varepsilon_k = \tan^{-1} 2^{-k}$ $k \geq 0$ $K = \prod_{k=0}^{n} \cos \varepsilon_k$	$x_0 = K$, $y_0 = 0$, $z_0 = \theta$ implies $x_{n+1} \approx \cos \theta$ $y_{n+1} \approx \sin \theta$	x_0, y_0 given, $z_0 = 0$ implies $z_{n+1} \approx \tan^{-1}(y_0/x_0)$ $x_{n+1} \approx (x_0^2 + y_0^2)^{1/2}/K$
$m = -1$ $\varepsilon_k = \tanh^{-1} 2^{-k}$ $k \geq 1$ $K' = \prod_{k=1}^{n} C_k$ $C_1 = \cosh \varepsilon_1$ $C_k = \cosh^2 \varepsilon_k$	$x_1 = K'$, $y_1 = 0$, $z_1 = \theta$ implies $x_{n+1} \approx \cosh \theta$ $y_{n+1} \approx \sinh \theta$ $x_{n+1} + y_{n+1} \approx e^{\theta}$	$y_1 < x_1$ given, $z_1 = 0$ implies $z_{n+1} \approx \tanh^{-1}(y_1/x_1)$ $x_{n+1} \approx (x_1^2 - y_1^2)^{1/2}/K'$
		$x_1 = w + 1$, $y_1 = w - 1$, $z_1 = 0$ implies $z_{n+1} \approx \frac{1}{2} \ln w$
		$x_1 = w + \frac{1}{4}$, $y_1 = w - \frac{1}{4}$, $z_1 = 0$ implies $x_{n+1} \approx w^{1/2}/K'$

Table 1 CORDIC Scheme

The mathematical rigor needed to justify convergence of each of these sequences to their desired value was alluded to by Volder; however, Walther was the first one to prove the following theorem. I found the proof in Schelin's article easier to follow than Walther's.

<u>*Theorem:*</u> Suppose $\varepsilon_0 \geq \varepsilon_1 \geq \varepsilon_2 \geq \ldots \geq \varepsilon_n > 0$ is a finite sequence of real numbers such that

(2) $$\varepsilon_k \leq \varepsilon_n + \sum_{j=k+1}^{n} \varepsilon_j \,, \qquad 0 \leq k \leq n,$$

and suppose r is a real number such that

(3) $$|r| \leq \sum_{j=0}^{n} \varepsilon_j \,.$$

If $s_0 = 0$ and $s_{k+1} = s_k + \delta_k \varepsilon_k$ for $0 \leq k \leq n$ where

$$\delta_k = \begin{cases} 1 & r \geq s_k \\ -1 & r < s_k \end{cases},$$

then

$$|r - s_k| \leq \varepsilon_n + \sum_{j=k}^{n} \varepsilon_j \,, \qquad 0 \leq k \leq n,$$

and, in particular,

$$|r - s_{n+1}| \leq \varepsilon_n \,.$$

The CORDIC scheme appears to have been developed to compute the sine and cosine function values. However, since the cases m = 0 and m = 1 have been discussed at these meetings by Prof. Bruce Edwards of the University of Florida, this presentation will focus on the m = -1 case. We must show that the inequality (2) is satisfied, indicate the relevance of the inequality (3), and show that the sequence does converge to the desired values.

m = -1

Let $\varepsilon_k = \tanh^{-1} 2^{-k} = \frac{1}{2}\ln\left(\frac{1+2^{-k}}{1-2^{-k}}\right)$ for $k \geq 1$. The inequality (2) is not satisfied; specifically, if k = n-1, it can be shown that $\varepsilon_{n-1} > \varepsilon_n + \varepsilon_n = 2\,\varepsilon_n$ but $\varepsilon_{n-1} \leq \varepsilon_n + 2\varepsilon_n = 3\,\varepsilon_n$. That is, the inequality (2) is satisfied for k = n-1 if the ε_n in the sum is repeated. Walther points out that certain steps in the iteration (1), and hence the corresponding ε_k, must be repeated. I found that the steps that needed repeating depended upon the choice of n. For example, if n = 13 then k=13 and k=4 need to be repeated whereas if n = 14 then k = 14, k = 5, and k = 2 need to be repeated in order to satisfy inequality (2).

We can get around the issue of which ε_k should be repeated by repeating all of them for $k \geq 2$. The proof of this is quite simple; mathematical induction is used. Let S_k be the statement

(5) $\quad S_k$: $\varepsilon_{n-k} \le \varepsilon_n + 2\sum_{j=n+1-k}^{n} \varepsilon_j$ is true for k = 1, 2, …, n-1.

If k = 1, then we want to show that $\varepsilon_{n-1} \le \varepsilon_n + 2\varepsilon_n = 3\varepsilon_n$. This is equivalent to

$$0 \le 3\varepsilon_n - \varepsilon_{n-1} = \frac{1}{2}\ln\left\{\left(\frac{1+2^{-n}}{1-2^{-n}}\right)^3\left(\frac{1-2^{n-1}}{1+2^{n-1}}\right)\right\}$$

which is equivalent to

$$(1+2^{-n})^3(1-2^{n-1}) \ge (1-2^{-n})^3(1+2^{n-1}).$$

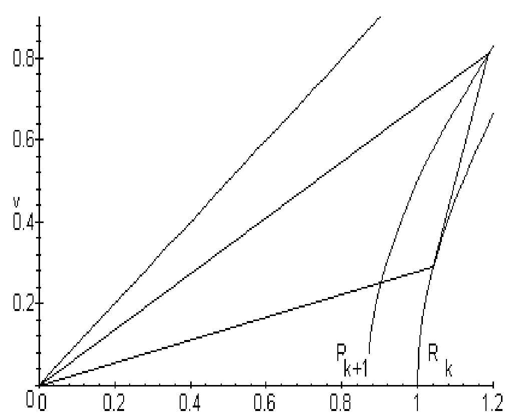
It is a simple exercise to show that this is true. Now assume S_k is true. We want to show that S_{k+1} is true. Replacing k with k+1 in (5) and taking all terms to the right side of the inequality to be shown, we have

$$0 \overset{?}{\le} \varepsilon_n + 2\sum_{j=n-k}^{n}\varepsilon_j - \varepsilon_{n-k-1} = \left\{\varepsilon_n + 2\sum_{j=n+1-k}^{n}\varepsilon_j - \varepsilon_{n-k}\right\} + \{3\varepsilon_{n-k} - \varepsilon_{n-k-1}\}.$$

The first term on the right is nonnegative by the induction hypothesis. The second term on the right can be shown to be nonnegative by repeating the steps performed in the S_1 case. Thus, S_k is true for all k = 1, 2, …, n-1.

The pictures I have seen describing the process for the m = -1 case did not help my understanding of the process. The picture at the right is similar to that found in the paper by Schmid and Bogacki.

The Case m = -1



The CORDIC scheme in the m = -1 case has recursion equations

$$\text{(6)}\qquad \begin{cases} x_{k+1} = x_k + \delta_k y_k 2^{-k}, \\ y_{k+1} = y_k + \delta_k x_k 2^{-k}, \\ z_{k+1} = z_k - \delta_k \varepsilon_k, \end{cases}$$

for $k \ge 1$ where $\varepsilon_k = \tanh^{-1}2^{-k}$.

Since $\tanh\varepsilon_k = \dfrac{\sinh\varepsilon_k}{\cosh\varepsilon_k} = 2^{-k}$

the hyperbolic identity $1 = \cosh^2 x - \sinh^2 x$ implies that

(7) $\quad \cosh\varepsilon_k = (1 - 2^{-2k})^{-1/2}, \quad \sinh\varepsilon_k = 2^{-k}(1 - 2^{-2k})^{-1/2}$.

Let $\theta_k = \theta_1 + z_1 - z_k$ where θ_1 is to be determined. Then

(8) $\quad \theta_{k+1} = \theta_k + \delta_k\varepsilon_k$

for each iteration of the scheme (6). Define R_k and θ_k so that $x_k = R_k\cosh(\theta_k)$ and $y_k = R_k\sinh(\theta_k)$. It follows from the CORDIC scheme and (8) that

(9) $\quad R_{k+1} = (1 - 2^{-2k})^{1/2}R_k = R_k/\cosh\varepsilon_k, \quad k \ge 1$,

for each iteration of (6) where R_1 is yet to be chosen depending on the function to be evaluated. Thus, θ_1 is chosen so that

$x_1 = R_1\cosh(\theta_1)$, $y_1 = R_1\sinh(\theta_1)$.

It is left as an exercise for the student to show that (7), (8), and (9) imply that the CORDIC scheme is satisfied.

We have already noted that (6) will be iterated twice for all $k \geq 2$ and, therefore, (9) implies that

$$R_{k+1} = (1-2^{-2})^{1/2}\left\{\prod_{j=2}^{k}(1-2^{-2j})\right\}R_1$$

Since we repeat iterations for $k \geq 2$ let us denote the first iteration with primes; that is,

$x'_{k+1} = x_k + \delta'_k 2^{-k} y_k = R_{k+1}\cosh(\theta'_{k+1})$,
$y'_{k+1} = y_k + \delta'_k 2^{-k} x_k = R_{k+1}\sinh(\theta'_{k+1})$,
$z'_{k+1} = z_k - \delta'_k \varepsilon_k$.

The second iteration is given by

$x_{k+1} = x'_{k+1} + \delta_k 2^{-k} y'_{k+1} = R_{k+1}\cosh(\theta_{k+1})$,
$y_{k+1} = y'_{k+1} + \delta_k 2^{-k} x'_{k+1} = R_{k+1}\sinh(\theta_{k+1})$,
$z_{k+1} = z'_{k+1} - \delta_k \varepsilon_k$.

Consider the rotation mode. Since $y_1 = 0$ it follows that $\theta_1 = 0$ and $x_1 = R_1$. Thus, the rotation mode assumes

$$x_1 = R_1 = (1-2^{-2})^{-1/2}\prod_{j=2}^{n}(1-2^{-2j})^{-1} = K',$$

$y_1 = 0$,
$z_1 = \theta$,
$z_{n+1} \approx 0$.

Then $\theta_{n+1} \approx \theta$, $R_{n+1} = 1$, and

$x_{n+1} \approx \cosh(\theta)$, $y_{n+1} \approx \sinh(\theta)$,

and, therefore,

$x_{n+1} + y_{n+1} \approx e^{\theta}$.

For what values of θ is CORDIC directly applicable and how does one get around this constraint? Since $z_{n+1} \approx 0$, $z_1 = \theta$, and $z_{n+1} = z_1 - \delta_1\varepsilon_1 - \Sigma(\delta_j + \delta'_j)\varepsilon_j$, we must have $|\theta| \leq \varepsilon_1 + 2\Sigma\varepsilon_j$. But $\varepsilon_1 + 2\varepsilon_2 > 1.0$. Therefore, convergence is guaranteed for $|\theta| \leq 1.0$. For arbitrary θ we repeatedly add or subtract ln(2) to get $\theta' = \theta - p\ln(2)$ where $|\theta'| \leq 1.0$. CORDIC is then applied to get

$\cosh\theta' \approx x_{n+1}$ and $\sinh\theta' \approx y_{n+1}$.

It follows from hyperbolic function identities that

$\cosh\theta \approx ½\,(x_{n+1} + y_{n+1})2^{p} + ½\,(x_{n+1} - y_{n+1})2^{-p}$

and

$\sinh\theta \approx ½\,(y_{n+1} + x_{n+1})\,2^{p} + ½\,(y_{n+1} - x_{n+1})2^{-p}$.

Now consider the vectoring mode. Since

x_1 and y_1 are given with $x_1 > |y_1|$,
$z_1 = 0$, and

$$y_{n+1} \approx 0,$$

it follows that $\theta_{n+1} \approx 0$ and, hence, $z_{n+1} \approx z_1 + \theta_1 = \theta_1$. Therefore, $y_1/x_1 = \tanh(\theta_1)$ and

$$z_{n+1} \approx \tanh^{-1}(y_1/x_1).$$

Moreover, since $R_{n+1} = R_1/K'$ in the general case,

$$x_{n+1} = R_{n+1}\cosh(\theta_{n+1}) \approx R_{n+1} = (x_1^2 - y_1^2)^{1/2}/K'.$$

There are two extensions of the vectoring mode. Since

$$\tanh^{-1} t = \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)$$

we set $t = y_1/x_1$ with $x_1 = w + 1$ and $y_1 = w - 1$ to get

$$z_{n+1} \approx \tfrac{1}{2}\ln(w).$$

Moreover, if $x_1 = w + \frac{1}{4}$ and $y_1 = w - \frac{1}{4}$, then

$$x_{n+1} \approx w^{1/2}/K'.$$

For what x_1 and y_1 can CORDIC be applied directly in the vectoring mode? Since $z_1 = 0$ implies $|z_{n+1}| \le \theta_1 + \varepsilon_1 + 2\Sigma\varepsilon_j$ and since $\varepsilon_1 + 2\varepsilon_2 > 1.0$ and $z_{n+1} \approx \tanh^{-1}(y_1/x_1)$, we require that $|\tanh^{-1}(y_1/x_1)| \le 1.0$. This is satisfied provided $|y_1/x_1| \le \frac{3}{4}$. Since, however, the domain of $\tanh^{-1}x$ is $|x| < 1$, we must deal with $\frac{3}{4} < |y_1/x_1| < 1$. Lastly, since $\tanh^{-1}x$ is an odd function, we shall assume $\frac{3}{4} < y_1/x_1 < 1$.

Walther points out in his paper that

$$\tanh^{-1}\left(1 - 2^{-E}M\right) = \tanh^{-1}(T) + \frac{E}{2}\ln(2) \tag{10}$$

where

$$T = \frac{2 - M - 2^{-E}M}{2 + M - 2^{-E}M}, \quad 0.5 \le M < 1, \text{ and } E \ge 1 \text{ integer.}$$

Thus, if $\frac{3}{4} \le y_1/x_1 = 1 - 2^{-E}M < 1$, then $0 < 2^{-E}M \le \frac{1}{4}$. The constraint $0.5 \le M < 1$ then implies

$$\frac{3}{11} \le \frac{7-4M}{7+4M} \le T \le \frac{2-M}{2+M} \le \frac{3}{5}.$$

Therefore, if $\frac{3}{4} \le y_1/x_1 < 1$ then we can choose $2^{-E}M = 1 - y_1/x_1$; that is, we obtain M be repeatedly multiplying $1 - y_1/x_1$ by 2 until we get

$$0.5 \le 2^E(1 - y_1/x_1) \equiv M < 1.$$

To compute $\tanh^{-1}T$ we use the given x_1 and y_1 to compute new values

$$x_1 \leftarrow 1 + M + y_1/x_1, \quad y_1 \leftarrow 1 - M + y_1/x_1$$

or

$$x_1 \leftarrow x_1 + y_1 + Mx_1, \quad y_1 \leftarrow x_1 + y_1 - Mx_1$$

which are now used in the CORDIC scheme. Then (10) is used to obtain $\tanh^{-1}(y_1/x_1)$.

Since $x_{n+1} \approx \sqrt{x_1^2 - y_1^2}/K'$ is playing the role of r in the theorem and since K' is approximately 1.25, the x_1 and y_1 must satisfy $\sqrt{x_1^2 - y_1^2} \le K' < 1.25$, we force the

$x_1^2 - y_1^2 \leq 1$ to apply CORDIC. If this condition is not satisfied, we repeatedly divide both x_1 and y_1 by 2 until their new values satisfy this condition. The desired value is obtained by multiplying the CORDIC solution by that power of 2.

References

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The Case m = -1