The Magic Calculator and The Sine Addition Formula

Students who have grown up with computers and calculators may take their capabilities for granted, but I find something magical about entering arbitrary values and computing transcendental functions such as the sine and cosine with the punch of a button. While the calculation seems like "white magic", the calculator operates like a "black box" - we don't teach our trigonometry students what happens inside. They must trust technology without knowing how it works or how they can independently check the results. However, this talk shows beginning trigonometry students **can** compute the sine and cosine of any angle to any desired degree of precision using only simple geometry and a calculator with a square root key.

Computing the Sines and Cosines

We will find the sine and cosine of an angle x which lies between two angles whose sine and cosine are known.

Finding Values Halfway Between Known Values. First suppose that x lies exactly halfway between the angles whose sine and cosine are known. Then the known angles are given by x+h and x-h for some angle h. As a first approximation we might try linear interpolation - i.e. averaging the known values of the sine and cosine for x+h and x-h. Geometrically, these average or interpolated values are the coordinates of the midpoint, A, of the chord joining the points P and Q corresponding to the angles x+h and x-h on the circle of radius 1 shown in figure 1.

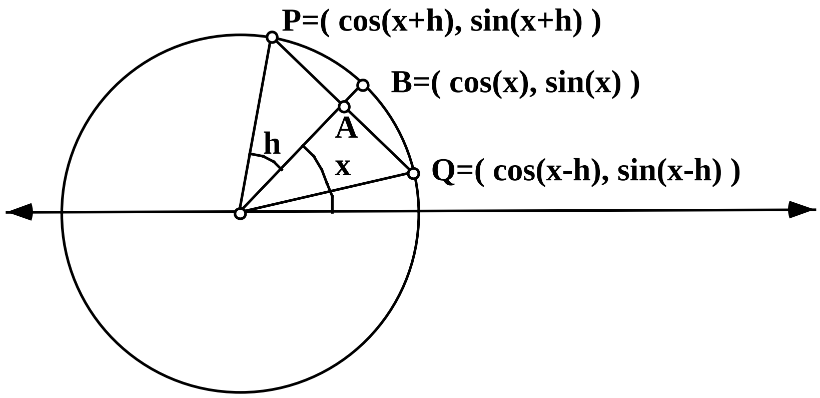

figure 1

So, the coordinates of A can be easily computed and used as an approximation to those of B, if A and B are not too far apart.

But we can do even better. We can use the coordinates of A and similar triangles to compute the coordinates of B exactly! (See figure 2.)

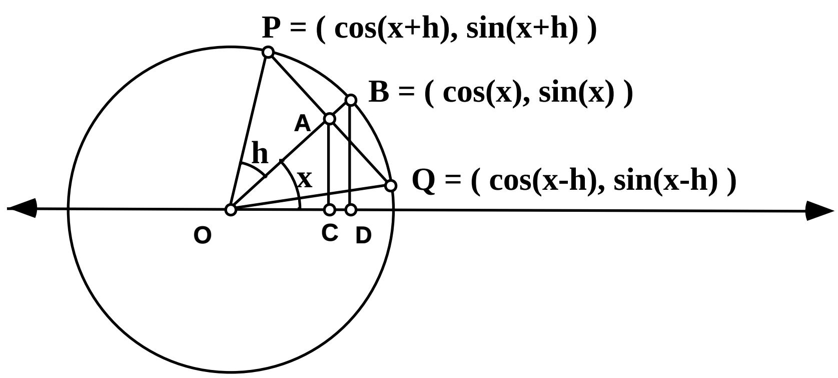

figure 2

Let $A=(a_1,a_2)$ where a_1 and a_2 are found by averaging the known coordinates of P and Q. Recalling that the radius of the circle is one, we can use similar triangles OAC and OBD to compute sin(x):

$$\sin(x) = \frac{\sin(x)}{1} = \frac{\sin(x)}{OB} = \frac{a_2}{OA} \quad , \qquad (1)$$

where $OA=\sqrt{a_1{}^2+a_2{}^2}$ can be computed using the Pythagorean theorem. Similarly

$$\cos(x) = \frac{a_1}{OA} \quad . \qquad (2)$$

Thus given the sine and cosine of any two angles, x+h and x-h, we can find the sine and cosine of the angle, x, halfway between them.

Finding Arbitrary Values Between Known Values. To find the sine and cosine of an arbitrary angle, y, located between two angles, x+h and x-h, whose sines and cosines are known, students can write and/or use a TI-82 program, (see handout: A TI-82 Program to Compute Sines from Scratch) that iterates the above procedure to "close in" on the desired angle. I have also had students use a spreadsheet to perform the computations one stage at a time. This method is similar to, and foreshadows, the "Midpoint Method" for finding the zeros of a continuous function that students may encounter in a later course.

Using the program, students find that the method "works" even if the known angles do not necessarily lie in the first quadrant and when sin(x) and/or cos(x) is negative. However, it gives unexpected results if the angle h is "too big". Explaining these results is valuable preparation for the derivation that follows, as we shall see.

Deriving sin(x+h) = sin(x) cos(h) + sin(h) cos(x)

I was very pleased to find that the diagrams used above also yield a simple derivation of this important identity. Oddly enough (bad pun intended), the only additional fact used is that sin(-x)=-sin(x).

First note that the length OA computed in the above procedure is cos(h). One way to see cos(h)=OA is to think of line OB as the horizontal reference line of the circle. Then angle h is in standard position and OA clearly gives the "horizontal" coordinate of P.

It turns out that substituting cos(h) for OA in formulas (1) and (2) above also makes them valid for cases where cos(h) is negative, i.e. when h is between 90 and 270 degrees (or coterminal with such an angle). Figure 3 shows this situation. We see that, when we divide the coordinates (a_1, a_2) by the length, L, of OA we would get the coordinates of the point on the circle opposite B. If we divide instead by -L (i.e. cos(h)), we get the correct values of cos(x) and sin(x). (This is the point where investigations of various values of x and h are helpful background.)

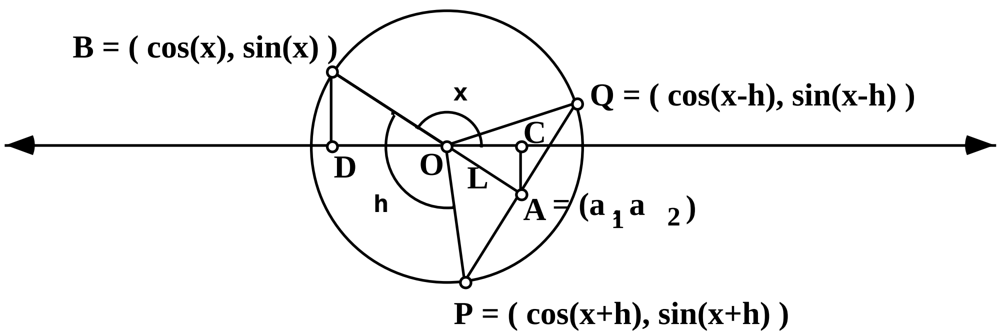

figure 3

Thus formulas (1), and (2) become:

$$\cos(x) = \frac{a_1}{\cos(h)} \quad (3) \qquad \text{and} \qquad \sin(x) = \frac{a_2}{\cos(h)} \quad . \quad (4)$$

These formulas are now valid for any values of x and h except when the denominator is zero. In this latter case, P and Q are on opposite sides of the circle, and $a_1 = a_2 = 0$. Even in this case the formulas:

$$\cos(x)\cos(h) = a_1 \quad (5) \qquad \text{and} \qquad \sin(x)\cos(h) = a_2 \quad (6)$$

obtained from (3) and (4) are valid. Thus (5) and (6) are valid for every choice of x and h.

Formula (6) opens the door to the angle sum identity for the sine function. From figures 1 or 3 and equation (6):

$$\frac{1}{2}(\sin(x+h)+\sin(x-h)) = a_2 = \sin(x)\cos(h) \qquad (7)$$

Since the above formula holds for every value of x and h, we may interchange the values in x and h to get $\frac{1}{2}(\sin(h+x)+\sin(h-x)) = \sin(h)\cos(x)$, or

$$\frac{1}{2}(\sin(x+h)-\sin(x-h)) = \sin(h)\cos(x) \qquad (8)$$

Adding (7) and (8) yields the desired formula:

$$\sin(x+h)= \sin(x)\cos(h) + \sin(h)\cos(x). \qquad (9)$$

Looking Back

I believe the derivation given here rivals the method found in most texts ([1] [2] [3] [4] [5] [6]) in logical simplicity. It's not logically necessary to relate formulas (5) and (6) to the computations of angles. They are a simple consequence of the similar triangles in figures 2 and 3, however I believe it is pedagogically useful.

Learning to compute sines and cosines of arbitrary angles provides motivation for the diagrams and students gain prior experience with the ideas ultimately used to obtain the formula. The student encounters and reconciles the difficult issues long before the identity is mentioned, assimilating the ideas over an extended period, rather than facing them all at once.

Besides the pedagogic utility of computing sine values, I strongly believe in the intrinsic value of providing an alternative to the acceptance of calculator output entirely on faith. I also like the way technology is used as a thread linking the numeric, geometric, and symbolic aspects of a meaningful computation, and especially how technology is used to explain how its own results might be obtained.

References

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