“TangentField”: A tool for “webbing” the learning of differential equations

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Principled design

“… being as explicit as possible about the space of possibilities and about the assumptions made in choosing from among them in the design process.”

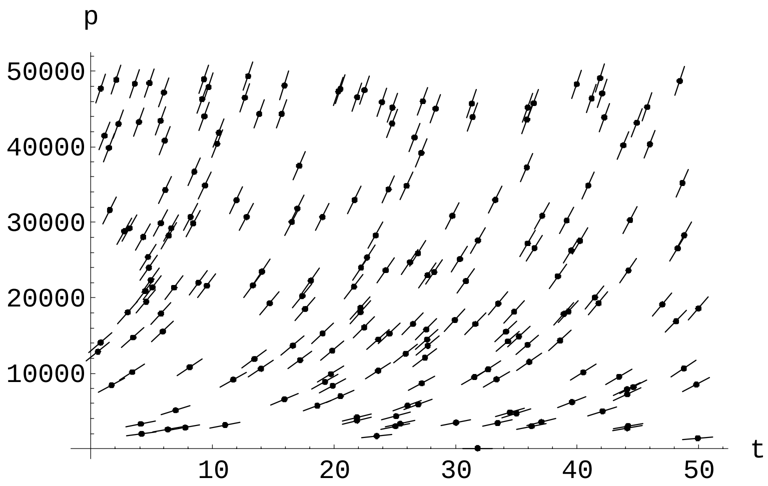
“… trying to articulate principles in the process of design can advance the state of the art by developing explicit and testable general ideas in a context where the actual impact of those ideas in selecting and generating design alternatives is visible.”

— Andrea diSessa (1985).

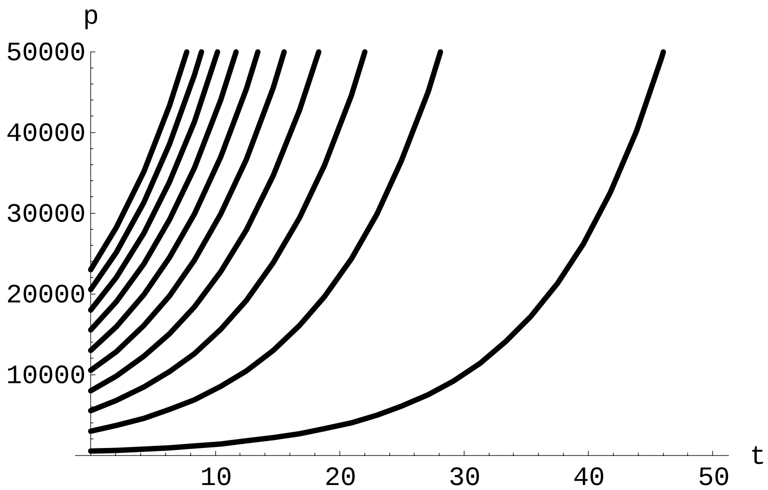
1. Differential equations and tangent fields

1.1 Representing differential equations

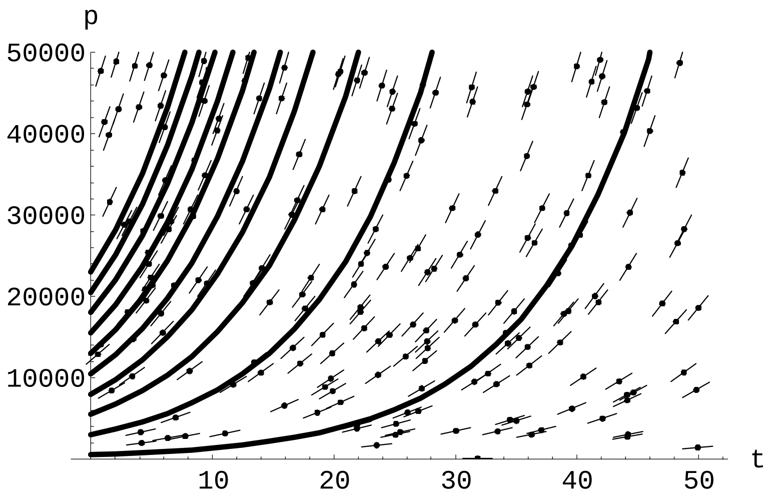
One way of representing a first-order differential equation graphically is by using a diagram in which a number of straight line stubs are placed on a graph, and each is tilted so that its *slope* is equal to the rate of change specified by the differential equation at that point. This is what we call a **tangent field**; it is often also called a “direction field”. Here is a field for the equation $\frac{dp}{dt} = 0.1p$:



The field is, of course, closely related to the set of graphs for the **family of curves**, representing the *solutions* of the differential equation. Here is a subset of those graphs:



The relationship can be seen more clearly if we show the tangent field and the family of curves on the same diagram:



1.2 What do you mean, “representing”?

To someone well versed in differential equations, the idea of “representing” such an equation by a tangent field is quite straightforward. But it is harder to say what this kind of diagram “represents” to a student who is entirely new to the topic and who has yet to build up a conception of “differential equation”.

This poster is about our attempt to build a computational tool, `TangentField`, based on the computer algebra system *Mathematica*, that allows students to use tangent fields not as a means of “representing” a concept with which they are already familiar, but as an aid to their learning of that concept.

2. Design and theory

We want to make explicit the key theoretical ideas that have influenced the design of `TangentField`, and the “ODE microworld” of which it forms a part (for a fuller description of the microworld as a whole, see Kent, James and Ramsden, in preparation). We give some specific examples of our attempts to instantiate these ideas at the micro-level of `TangentField`’s design and the learning activities which we wrote.

2.1 Webbing

Noss and Hoyles (1996) coined the term “webbing” to describe the process of learning in a globally structured and locally responsive learning environment. The global structure is partially established by the environment’s designers in terms of the computational tools provided and the activities in which the tools are to be used. But this global structure is not fixed: the tools are extensible through combination, and in principle each one is itself modifiable; likewise, activities are intended to be in part resources for learners’ own modifications and combinations. In this way, the “web” is extended and re-structured as a result of connections made by learners.

In developing `TangentField` we wanted a tool that would provide an entry into the concept of differential equation and a support to learners' development of that concept[1]. The support, as conceived by Noss & Hoyles, works by "embedding" in the tool both the hypothesised starting concepts of the learner, and the concepts we intend the learner to end up with.

The entry point we chose was suggested by the research literature: Dreyfus (1990) and Harel & Kaput (1991) describe research which indicates that many beginning calculus students tend to see the mathematical definition of *gradient* (slope) as applying at a single point. To develop a concept of differential equation one needs to see gradient as being defined everywhere along a curve, or even everywhere in a plane: the derivative is a function.

Thus, one of the inputs to `TangentField` is a list of coordinates. The input of a single coordinate is compatible with the starting concept of "pointwise derivative". But `TangentField` accepts a *list* of coordinates, thus the idea of derivative as a function (which is something defined for a set of points) is "embedded" in the tool.

When you use `TangentField`, you find you can see patterns of *curves* among the tangent stubs. An expert user can see these for what they are: the graphs of the *solutions* of the differential equation. But the learners do not need the concept of solutions to see these curves. We hoped that the learners, in constructing a meaning for these families of curves, would *through this* construct the concepts of "solutions" and "differential equation". For Noss and Hoyles, the crux of webbing is this *supported* structuring and restructuring of the learner's mathematical ideas.

Since our target learners are physical scientists (chemists and biochemists), the precise connections between differential equations as part of "computational mathematics" (webbed through tools like `TangentField`) and as part of "official" paper-and-pencil mathematics are not quite the point of concern for us. Our concern was, rather, to promote the learners' forging of links between the mathematics and the chemistry. We wrote two complementary sets of activities: one "mathematical" (the *Experiments*) and the other "contextualised"[2] in the topic of reaction kinetics, a very basic chemical situation where differential equations appear as components of models for rates of reactions. Whilst certainly the major concern of the students is to use mathematics in their chemistry, our hope was that their knowledge of chemistry would function as webbing support for learning about differential equations with `TangentField`. And this is what happened when we tried `TangentField` with students—see Example 2 (Section 3.2). In fact, those two students knew a lot more about reaction kinetics, and it was fresher in their minds, than probably they knew, or could clearly remember, about calculus *per se*.

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[1] Later on in this work, we became aware of David Tall's use of "direction fields" in his *Graphic Calculus* software, and his similar intentions of using a computational medium for "building on what students already know in a way which is consistent with their cognitive development" (Dubinsky and Tall 1991, p. 238; see also Tall 1989).

[2] We see "contextualised" as a problematic term. We are currently doing theoretical and design work to explore the relationships between "mathematics" and "contexts". (For some early ideas, see Templer et al, in press.)

2.2 The course of meaning-making

`TangentField` and its accompanying activities (for example, "Experiment 1", in the next section) form an extremely complex environment for learners to make meanings about mathematics. There are algebraic "texts" (e.g. the equations $dy/dx = \sin x$, $da/dt = -ka^2$) which are ambiguous, and there are graphical "texts" (the tangent fields, solution curves, and families of curves). We have used the term "text" rather than "representation" since one of the main points we wish to make is, what is a representation of what? In the writing of the `TangentField` activities we faced many choices about the steps in meaning-making that we intended a learner to construct. We'll give one example that we think is of central importance.

We expected that learners might come to our activities reading a text like "$dy/dx = 2x$" as if it were a part of a text with which they are very familiar from differentiation: "$y = x^2$, $dy/dx = 2x$". We wanted them to develop another reading for the text: "an equation with a family of solutions". In pencil-and-paper mathematics it is difficult to do anything other than to state "this expression is now to be read as an equation": a circular situation where the notion of differential equation must, in a sense, be assumed in order for a learner to make sense of such a statement.

At one point in the design process we reached the following situation: we'd already hit upon `TangentField` as a very rich tool for dealing with differential equations, and had made use of it in an activity to do with Euler's algorithm for numerical solution. And now we were asking, what sorts of tools and activities would serve to develop the idea of "differential equation", avoiding the conventional circularity? ... The realisation that `TangentField` was precisely the tool we needed came very suddenly and unexpectedly, and the reasons why we couldn't "see it" are interesting.

A basic aim of our ODE microworld design work was to get out of the well-worn paths of mathematics textbooks and to investigate what a computational medium could provide to "re-vision" the subject. Tangent fields are a well-known representation for differential equations, but to see `TangentField` as the way out of the conventional circularity involves making it central to the process of meaning-making for differential equations. In retrospect, we realise that we were constrained to see `TangentField` in its auxiliary, computational role, supporting the "official" representations of differential equations in pencil-and-paper mathematics. When the constraint lifted, we felt simultaneously rather silly, since re-visioning been our precise intention in using the computational medium in the first place, and also very excited, partly because whilst the task was to design a learning environment for learners, revisioning mathematics also means revisioning it for yourself. An old, familiar area of mathematics could be experienced *by us* in a different way, and *our* concepts of differential equation have been greatly enriched by using`TangentField`.

3. Students using TangentField in "Experiment 1"

On the following page we reproduce the first activity that the students are asked to attempt using `TangentField`.

Experiment 1

Solving first-order differential equations

Task: For each of the differential equations given in the "Mathematical objects" box, (i) plot a tangent field, (ii) solve the equation by hand to get the general solution, and (iii) solve it using the `DSolve` command. Check that the family of curves matches up with the corresponding tangent field.

Mathematical objects

$\frac{dy}{dx} = \sin x$; $\frac{da}{dt} = -ka^2$ (k is a constant); $\frac{dy}{dx} = ax^2$ (a is a constant);

$\frac{da}{dt} = -ka$ (k is a constant); $\frac{dy}{dx} = \frac{-4x}{9y}$; etc. …

IDEA: solution of separable differential equations

When the right-hand side of a DE can be written as a product of a function of x alone and a function of y alone, the general solution can be found directly by integration. If I can write $dy/dx = F(x)\,G(y)$, then:

$$\int \frac{1}{G(y)} dy = \int F(x) dx + c$$

For example, if $dy/dx = 2x$,

$$\int 1 dy = \int 2x dx + c$$

$$y = x^2 + c$$

***Mathematica* objects**

```
DSolve[y'[x] == 2x, y[x], x]
DSolve[y'[x] == -y[x], y[x], x]
ParametricPlot[{3 Cos[t], 2 Sin[t]}, {t, 0, 2Pi}]
```

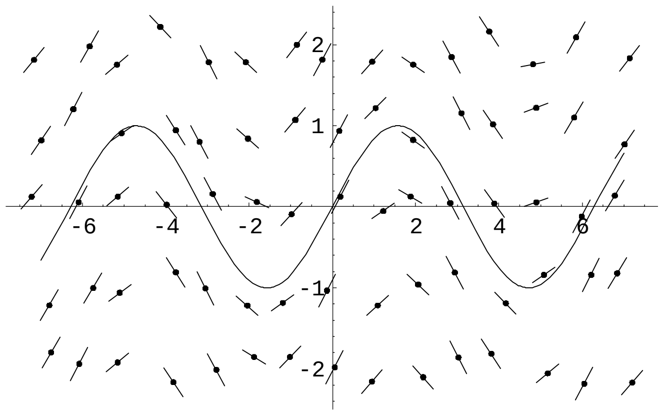
***Mathematica* functions**

```
DSolve, Plot, Show, TangentField, RandomGrid, ScatteredPoints
```

We present below three examples taken from an audio transcript of a two-hour observed session (and computer data was captured in a dribble file). The students involved were a pair of first-year Biochemistry undergraduates ("A" and "J"). They had been using *Mathematica* on-and-off for about 7 months, and had done some related activities on *Reaction Kinetics* that we wrote (the "contextualised" set of activities mentioned in Section 2.1). This was their first encounter with `TangentField`, and the first with differential equations as "mathematical" objects since being in school. We have made some annotations to the transcript in**—bold—**.

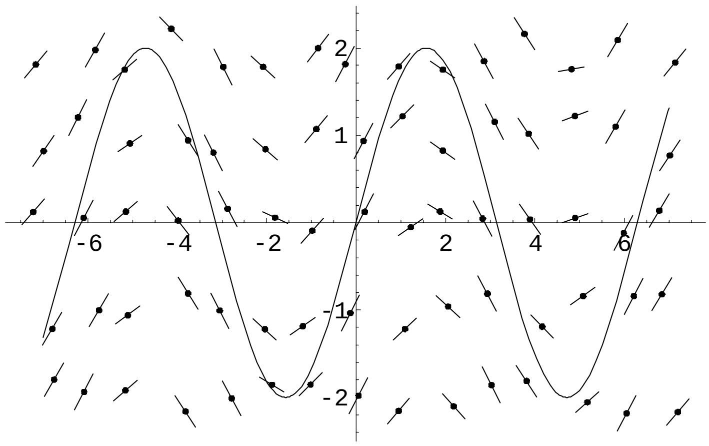
3.1 Example 1: "What do you think about this stuff up here though?"

A and J have produced a TangentField for cos(x), and a plot of sin(x), and are satisfied with what they've got:



Observer: What do you think about this stuff up here though? (*indicating the portion of the tangent field above the region of sin(x)*)

A: **—with no hesitation—**That's when you've got like two sine-x or sine two-x or something like that.... I think anyway... if we change this to whatever, this value here, make that two... (*changes sin(x) to 2 sin(x)*) ... and then if you plot that it should...



A: See sine-x was down here, this tangent here... and so...

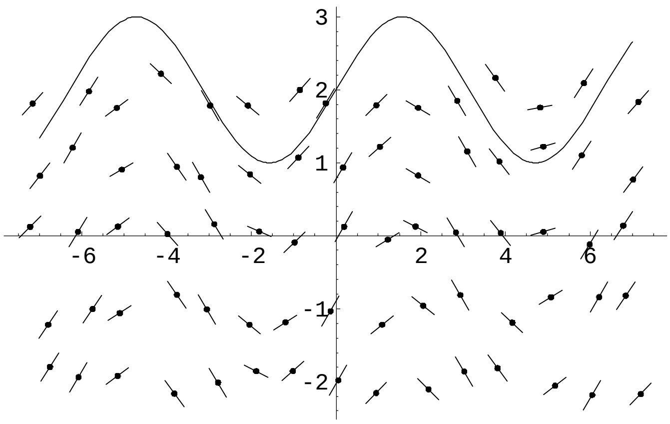
Observer: Uhm, I'm not convinced.

A: No because the derivative's changed up here hasn't it? what's the derivative of two sine-x?

Observer: What could you do where the derivative wouldn't change?

A: Er, plus something or another, plus c.

J: Plus a constant.



J: It’s just where it’s been shift[ed] up and down the graph isn’t it?

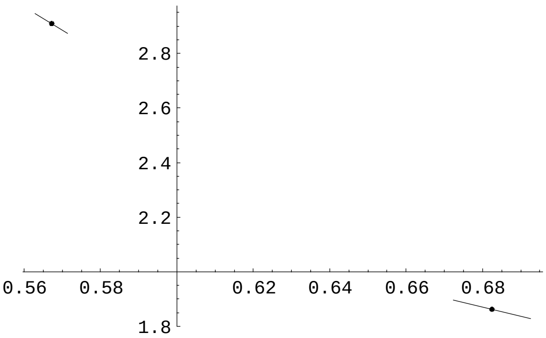
A: I like my original one...

Changes sin(x) + 2 back to sin(x).

—Overall, the students seemed to be interested in and engaging with activities. Here the students seemed to have minimal independent interest: their attitude was rather one of satisfying the Observer’s request—

3.2 Example 2: Webbing with chemical knowledge

When the students got to the second differential equation in the list of “mathematical objects”, $da/dt = -ka^2$*, they immediately began to draw upon their chemical knowledge about reaction kinetics, where such an equation is a model for the rate of change of concentration of a chemical (a) with respect to time (t) in a chemical reaction:*



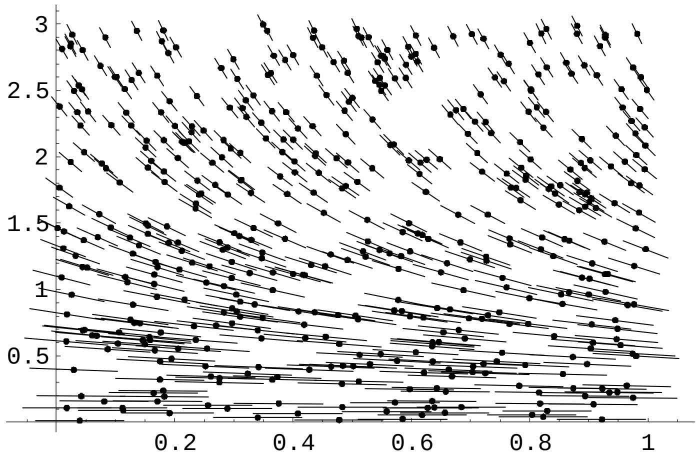
J: You can’t have minus time, so why is it going...

A: You can, because this isn’t radioactive decay is it?

J: But you can’t have minus time (pause) can you?

A: But who says it’s time? it’s just an equation... see? we’re just calling it t because we’re used to calling it time... we could call it p.

Eventually they get a TangentField plotted over the right plot ranges to be “meaningful”:



J: It's got that nice general shape, they're all coming to ... infinite origin (*movement of hand following general "shape").*

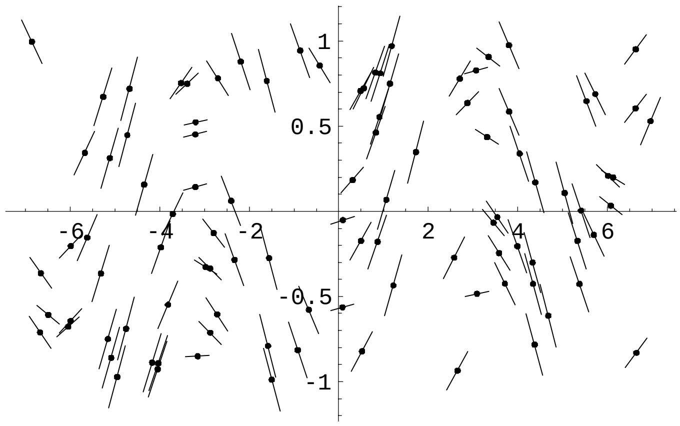
Observer: I see you what you mean when you said they're all coming like that, is that in chemistry?

A: Yes... radioactive...

J: They're all meant to come to sort of like eventually no matter what concentration of a you start with, in a kinetic reaction it's all meant to come to eventually one sort of end product, substrate concentration, the enzyme can only use, like it reaches, a point where the enzyme can only use so much ...

3.3 Example 3: Inverting the task

Using an example statement that we provided, A and J managed to produce, after a bit of bother with syntax, a TangentField for $dy/dx = \sin x$ *:*



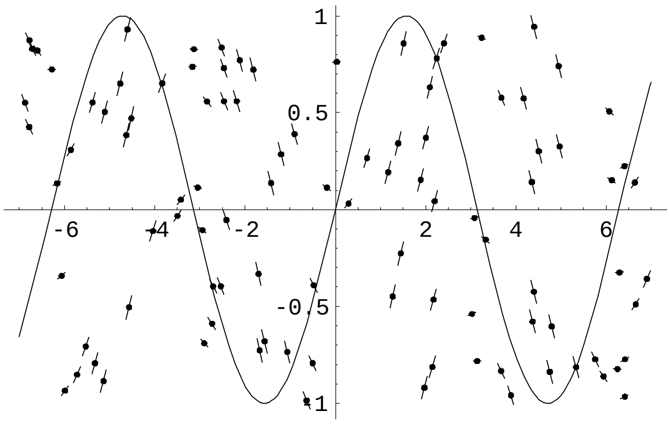
J: Oh my God, it's given the general shape though.

A: Yes we've got the general shape...

J: ... because you know what sine (inaudible: "function"?) looks like anyway.

Despite knowing that they're dealing with the equation $dy/dx = \sin x$ *, what is in their minds (as emerges explicitly later) is that TangentField is drawing the tangents to the curve y = sin(x). Hence after playing about with changing the tangent lengths for a bit:*

A: ... why don't we try plotting sine-x on the same graph? (*they do so...*)



J: ... right, now what have we got to do? ... (reads) plot a tangent field—we've done that.

A: (reads) Solve the equation by hand to get the general solution, and solve it using dsolve.

Observer: Can I take you to something?

Both: Yep.

Observer: Here's your delightful parabolas and tangent fields, yes? (*the example we provided was for* $dy/dx = 2x$)... now look at what you've got... I can't say I'm thoroughly convinced.

Both: (laughter) Nor am I!

Observer: This is a tangent field and you can see it's tangent... (*in the given example*)

J: Yes... they're not (*referring to their graphs*).

A: They're not that tangent.

J: They seem to be shifted over don't they? **—Very interesting and undeveloped remark!—**

A: OK, so what's gone wrong up here? ... maybe it's not the scattered point then, I don't think.

Their first reaction is to question how they're selecting coordinate points. ***—As if they're appearing in 'the wrong places'??—*** *This does not prove productive and some time later:*

A: Let's have a read, let's see what it says again... ... ah hah!... it's plotting the, tangent field plots the derivative, now we've got to work out the derivative of sine-x before we can use tangent field.

J: Cos of x, minus-cos-x... no hang on, no no it's cos-x, cos-x goes to minus-sine-x.

They produce a TangentField for cos(x), and a graph for sin(x), and try out different plotranges and tangent lengths. They're happy that the "tangents are following the shape". Later, after the episode of Example 1, the Observer intervenes:

Observer: Can I just ask you, do you think you've solved *that*? (*indicates "dy/dx = sin(x)" in Experiment 1*)

A: Er, no.

Observer: You've done the opposite way round haven't you?

J: Yes, we solved it and then fitted it to it (laughs!).

A: We fiddled it. OK maybe we should do the dsolve bit.

Observer: Yes you've said if my solution's sine-x what will my derivative be? ... whereas in fact here the derivative's sine.

A: Oh, we're so silly aren't we? (laughs)

Observer: But, I mean, it's all much of a muchness isn't it?! (laughs)**—Observer colludes in the inversion of the task—**

A: Same principle.

J: So basically we should have started with minus-cos-x.

4. Concluding remarks: TangentField—what's in a name?

Having seen students using the tool, we now think that the name `TangentField` may be problematic.

We wanted the students to make a meaning for the family of curves that emerge as `TangentField` is used and thus make a meaning for a relation like $dy/dx = \sin x$ as a differential equation. We were hoping that students might be surprised at the emergence of families of curves and thus encouraged to supply them with meaning.

But our interpretation of the session transcript is that the students were in some sense expecting a multiplicity of curves. The trouble is, they did not seem to be thinking in terms of *families of solutions*, but rather "the curves that have these tangents".

The `Tangent` in `TangentField` brings this interpretation to mind: if "tangent" then "tangent to what?" We were hoping that the students would "fit" solutions to a tangent field, but we found that the question the students chose to answer was "what curve, when I differentiate it, has this tangent field?"—see Example 3.

Of course, this is one way of viewing a differential equation, but we suspect that it is one that marginalises the concept of *family* of solutions and perhaps does not even require the algebraic relation to be interpreted as an equation. Indeed, the students appeared to consider their task completed if they found a *single* curve that fitted the field, in spite of being asked explicitly to find families of solutions.

They would, upon intervention from one of us, quite happily "move around" their single curve by the appropriate use of a constant—see Example 1—this was not surprising to them but neither, unfortunately, was it interesting to them.

So we are going to rename the tool. Perhaps `GradientField`? We would be delighted if anyone has any other suggestions, or would like to comment on whether they think that `Gradient` will give rise to the same problems as `Tangent`. And should we be calling it a `Field` at all?

As the three examples, and the above discussion make clear, the design choices that we made for the first version of the ODE microworld were certainly not flawless. But the unfortunate effect of `TangentField`'s name serves to confirm that particular choices really do matter. We are struck by the acuteness of diSessa's (1985) remark about *principled design*: "being as explicit as possible about the space of possibilities and about the assumptions made in choosing from among them in the design process". Being explicit about possibilities and about assumptions is no mean task: consider, for example, our surprise at being able to re-vision differential equations with `TangentField`, and our failed assumptions about the name (that learners would find it as informative and unambiguous as we did). But diSessa's other remark (quoted on page 1) points to a systematic way forward: *trying* to articulate principles in the process of design (in our case a theoretical framework to analyse meaning-making), *can* advance our work by providing ideas that are *testable* in the context of design choices. In testing our designs and closely examining the results, we can improve the designs *and* advance the state of the theoretical framework.

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* Available from `http://metric.ma.ic.ac.uk/articles/` .