Poster/Paper

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Title:

Calculator/Geometry Strategies for Understanding Properties of $x^{m/n}$.

Abstract:

Some students may benefit in the understanding and applying real numbers of the form $x^{m/n}$ from a combination of calculator and theoretical geometric techniques. The poster/paper will demonstrate a seven-step calculator routine for expressing $x^{m/n}$ and a spiral sequence of similar right trigons for conceptualizing $x^{m/n}$. Further calculator routines and trigonal spirals will demonstrate properties relating $x^{m/n}$ and $x^{M/N}$. Analysis of such trigonal spirals will reveal strategies for exploring comparisons of $x^{m/n}$ and $x^{M/N}$. Finally, a summary analysis for the theoretical foundations of each spiral will be included.

Poster/Paper:

Quantum jumps in cognitive development/maturation with respect to understanding and applying real numbers seem to occur in transitions from integers to rational numbers and from rational numbers to irrational numbers. The focus of this poster/paper is upon enhancing the transition from rational to irrational numbers, particularly, irrational numbers expressible as $x^{m/n}$ and comparing $x^{m/n}$ and $x^{M/N}$. The specific emphasis is upon assisting those students whose learning style compliments the combination of using a calculator and theoretical geometrical models.

The calculator technique for representing $x^{m/n}$ is a simple seven-step routine; $C\left(x^{m/n}\right)$: ON, x, y^x, m, $/$, n, $=$ ____ ; which renders a decimal approximation for $x^{m/n}$ (a seemingly rational expression). The calculator strategy for comparing $x^{m/n}$ and $x^{M/N}$ would be to use the seven-step routine to decimally express $x^{m/n}$ and $x^{M/N}$ and then compare the decimal expressions which are approximations. This strategy may be somewhat satisfying even though such approximations and observations could possibly lead to overly simplified results of analysis and could possibly lead to erroneous conclusions (such as $10^{1/1000}$ and $10^{1/999}$ are each approximately 1.00231 on a calculator with a seven-digit display). The calculator as an educational tool may be satisfying for some students in isolation, but is probably not satisfying, by itself, for all students.

A theoretic geometric approach for comprehending $x^{m/n}$, as proposed in this paper, is a sequence of similar right trigons which construct a spiral (perhaps like an unfolding fan of right trigons). The sequence of similar trigons is constructed by requiring the terminal edge of a preceding trigon to be congruent and coincident with the initial edge of the subsequent trigon. The initial and terminal edges of a trigon are mutually perpendicular and are each measured by a ruler. The first trigon of the sequence has an initial edge of one unit and a terminal edge of x units (for the purposes of this paper x is greater than one). A typical example of such a sequence or spiral (or fan) is depicted in the first diagram.

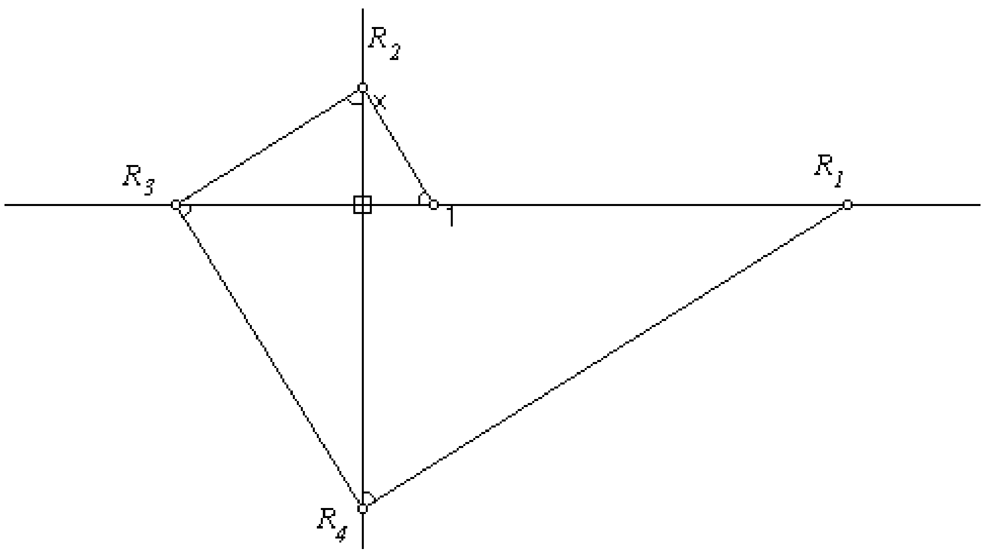


Diagram 1. A typical spiral of similar trigons.

Similar to the observation about calculator results in isolation, although this geometric strategy in isolation may be somewhat satisfying to some students, this strategy is probably not satisfying, by itself, for all students.

The poster on display is basically composed of five parts:

A large Central Display which graphically represents three spirals, $2^{n/1}$, $2^{n/2}$ and $2^{n/3}$, with corresponding ruler coordinates and calculator routines, and relates $\left(2^{0/1}, 2^{0/2}, 2^{0/3}\right), \left(2^{2/1}, 2^{6/3}\right), \left(2^{4/1}, 2^{8/2}, 2^{12/3}\right)$;

An Upper Left Display which exhibits two spirals, $3^{n/2}$ and $3^{n/4}$, and geometrically relates the pairs $\left(3^{0/2}, 3^{0/4}\right)$ and $\left(3^{4/2}, 3^{8/4}\right)$ with corresponding ruler coordinates and calculator routines;

An Upper Right Display which displays two spirals, $4^{n/3}$ and $4^{n/6}$, such that the ruler system for the second spiral is rotated $\frac{1\pi}{2}$ radians to relate $\left(4^{3/3}, 4^{6/6}\right)$ with corresponding ruler coordinates and calculator routines;

A Lower Left Display which graphs two spirals, $5^{n/3}$ and $5^{n/6}$, such that the ruler system for the second spiral is rotated $\frac{2\pi}{2}$ radians to relate $\left(5^{2/3}, 5^{4/6}\right)$ with corresponding ruler coordinates and calculator routines;

A Lower Right Display which displays two spirals, $6^{n/4}$ and $6^{n/8}$, such that the ruler system for the second spiral is rotated $\frac{3\pi}{2}$ radians to relate $\left(6^{1/4}, 6^{2/8}\right)$ and $\left(6^{5/4}, 6^{10/8}\right)$ with corresponding ruler coordinates and calculator routines.

Copies of the five displays are attached.

A result of constructing spirals for representing $x^{m/n}$ and for relating $x^{m/n}$ and $x^{M/N}$ is an analysis of ways of using spiral displays and/or the corresponding ruler coordinates, other than merely "guess and test" techniques. Rather than graphically representing several spirals on one coordinate system to make specific comparisons of $x^{m/n}$ with $x^{M/N}$, one can represent individual $x^{m/n}$ on transparencies and rotate pairs of transparencies to make specific comparisons on a geometric basis. Another strategy is to tabulate the coordinate systems separately on transparent sheets and shift sheets vertically to help locate appropriate pairs arithmetically and then determine how to orient the geometric spiral graphs on transparent sheets. Faculty and students can become proficient in either or both techniques readily and compliment such visual comparisons with calculator results, as well.

The sequence of similar right trigons, a spiral, with an initial edge of one unit yields a sequence of: $1st\Delta$, $2nd\Delta$, $3rd\Delta$, $\ldots$, $nth\Delta$,

$$\frac{1}{x^1} = \frac{x^1}{x^2} = \frac{x^2}{x^3} = \ldots = \frac{x^{n-1}}{x^n},$$

and is easily demonstrated by mathematical induction to yield the fact that the terminal edge of the *n*th trigon is x^n. If $x^n = y$, then $x = y^{1/n}$. Consequently, a sequence of terminal edges of the spiral constructed of similar right trigons is: $y^{0/n}$, $y^{1/n}$, $y^{2/n}$, …, $y^{n/n}$, …, $y^{m/n}$, …

The author, throughout his professional career, has continually searched for models to enhance student conceptualization of numerical ideas, as well as other mathematical ideas. The author has been well aware of traditional techniques for modeling square roots. A couple of years ago, the author became aware of a sequence of similar trigons which generated a sequence of spirals which were useful to the ancient Greeks (about 300 BC) in designing war machines which required irrational measures. The author has reinterpreted the geometric situation to generate a sequence of similar right trigons (which seems to be more easily understood in our tradition) which in turn generates a sequence of spirals for representing $x^{m/n}$.

CENTER DISPLAY: Three spirals, $2^{n/1}$, $2^{n/2}$, $2^{n/3}$, with a common initial ruler, $R_1 = R'_1 = R''_1$, demonstrating $2^{0/1} = 2^{0/2} = 2^{0/3}$; $2^{2/1} = 2^{6/3}$; and $2^{4/1} = 2^{8/2} = 2^{12/3}$.

Graphics:

$2^{n/1}$ (Thick line) $2^{n/2}$ (Thin line)

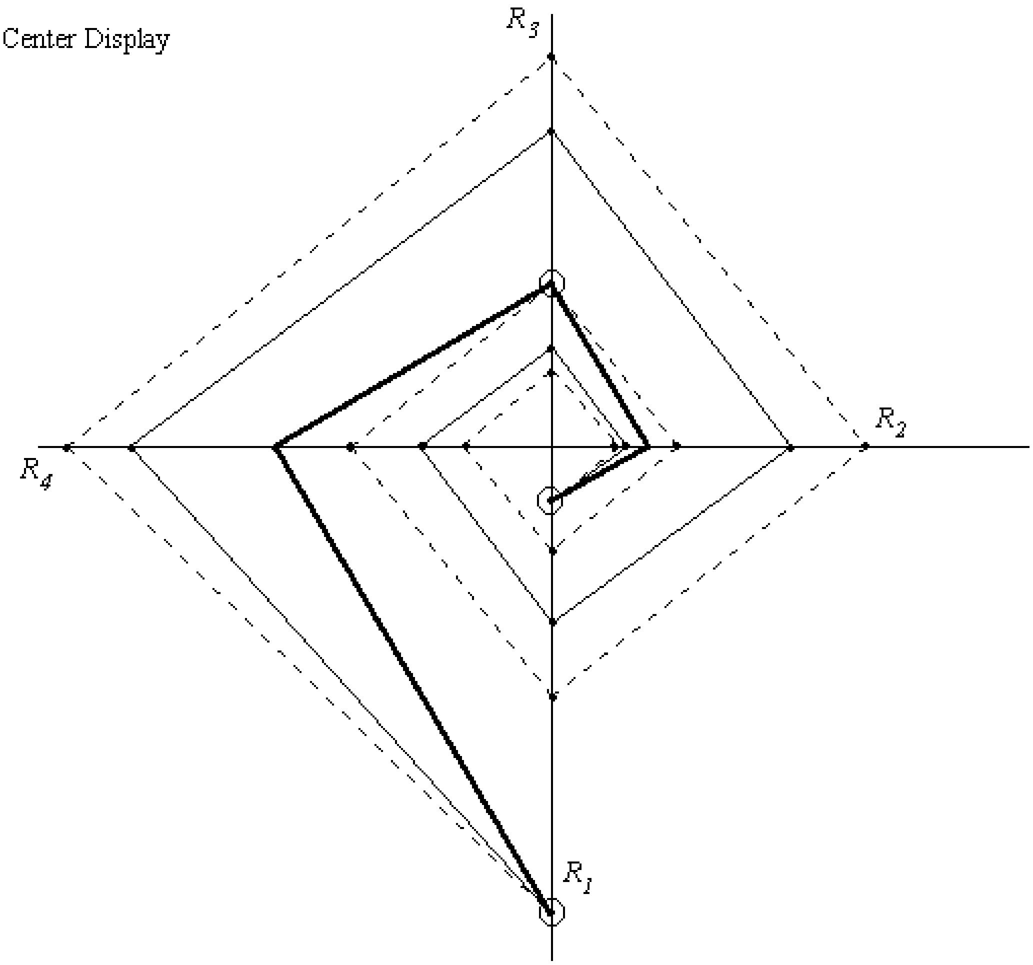
$2^{n/3}$ (Dashed)

*	R_1	$2^{0/1}$	*	R'_1	$2^{0/2}$	*	R''_1	$2^{0/3}$
	R_2	$2^{1/1}$		R'_2	$2^{1/2}$		R''_2	$2^{1/3}$
**	R_3	$2^{2/1}$		R'_3	$2^{2/2}$		R''_3	$2^{2/3}$
	R_4	$2^{3/1}$		R'_4	$2^{3/2}$		R''_4	$2^{3/3}$
***	R_1	$2^{4/1}$		R'_1	$2^{4/2}$		R''_1	$2^{4/3}$
	R_2	$2^{5/1}$		R'_2	$2^{5/2}$		R''_2	$2^{5/3}$
	R_3	$2^{6/1}$		R'_3	$2^{6/2}$	**	R''_3	$2^{6/3}$
	R_4			R'_4	$2^{7/2}$		R''_4	$2^{7/3}$
	R_1		***	R'_1	$2^{8/2}$		R''_1	$2^{8/3}$
	R_2			R'_2	$2^{9/2}$		R''_2	$2^{9/3}$
	R_3			R'_3	$2^{10/2}$		R''_3	$2^{10/3}$
	R_4			R'_4			R''_4	$2^{11/3}$
	R_1			R'_1		***	R''_1	$2^{12/3}$
	R_2			R'_2			R''_2	$2^{13/3}$
	R_3			R'_3			R''_3	$2^{14/3}$

Calculator Routines:

$C(2^{n/1})$: ON, 2, y^x, n, /, 1, = ___ ;
$C(2^{n/2})$: ON, 2, y^x, n, /, 2, = ___ ;
$C(2^{n/3})$: ON, 2, y^x, n, /, 3, = ___

Center Display



UPPER LEFT DISPLAY: Two spirals, $3^{n/2}$ and $3^{n/4}$ are displayed such that the initial ruler, R'_1, of the second spiral, $3^{n/4}$, coincides with the initial ruler, R_1, of the first spiral, $3^{n/2}$. That is, the system for the second spiral has been rotated $\frac{0\pi}{2}$ radians. This display demonstrates $3^{0/2} = 3^{0/4}$, and $3^{4/2} = 3^{8/4}$.

Graphics:

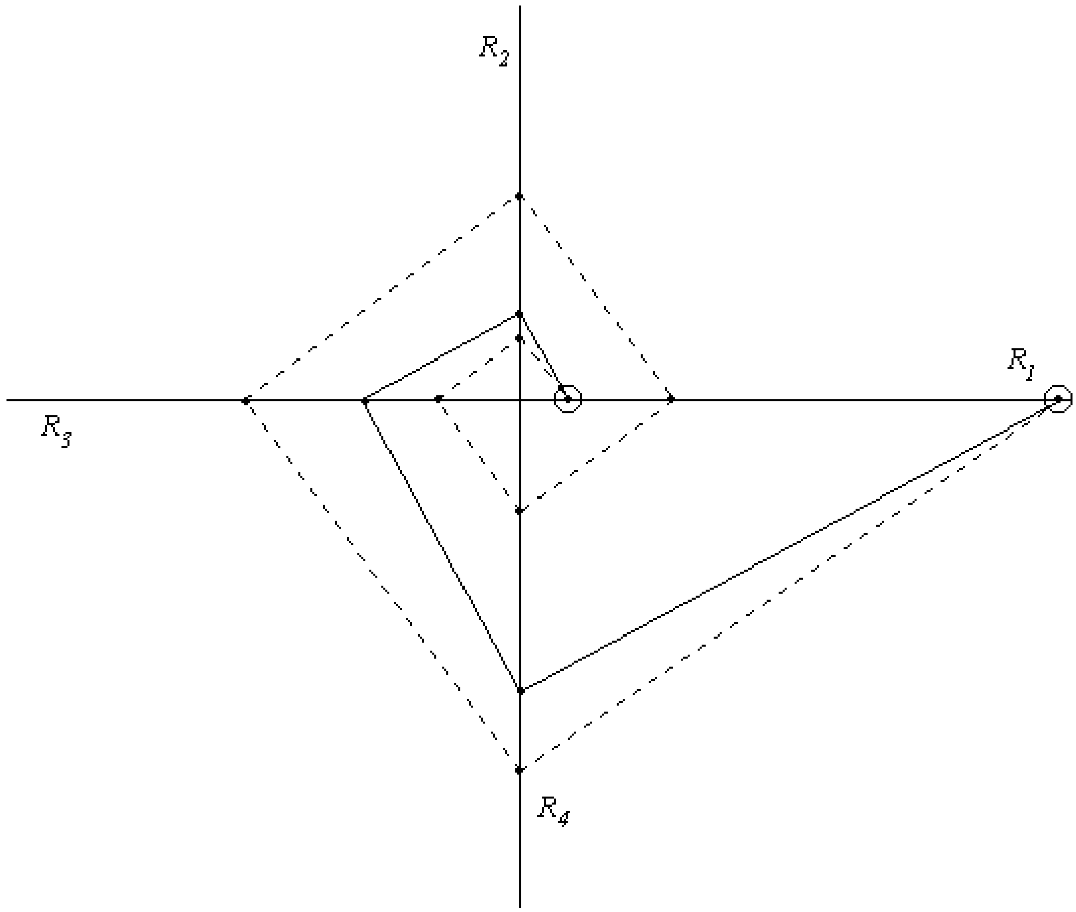
		$3^{n/2}$	(Solid line)			$3^{n/4}$	(Dashed)
*	R_1	$3^{0/2}$		*	R'_1	$3^{0/4}$	
	R_2	$3^{1/2}$			R'_2	$3^{1/4}$	
	R_3	$3^{2/2}$			R'_3	$3^{2/4}$	
	R_4	$3^{3/2}$			R'_4	$3^{3/4}$	
**	R_1	$3^{4/2}$			R'_1	$3^{4/4}$	
	R_2				R'_2	$3^{5/4}$	
	R_3				R'_3	$3^{6/4}$	
	R_4				R'_4	$3^{7/4}$	
	R_1			**	R'_1	$3^{8/4}$	

Calculator Routines:

$C(3^{n/2})$: ON, 3, y^x, n, /, 2, = ___

$C(3^{n/4})$: ON, 3, y^x, n, /, 4, = ___

Upper Left Display



UPPER RIGHT DISPLAY: Two spirals, $4^{n/3}$ and $4^{n/6}$, are displayed such that the initial ruler, R'_1, of the second spiral, $4^{n/6}$, coincides with the second ruler, R_2, of the first spiral, $4^{n/3}$. That is, the system for the second spiral has been rotated $\frac{1\pi}{2}$ radians. This display demonstrates $4^{3/3} = 4^{6/6}$.

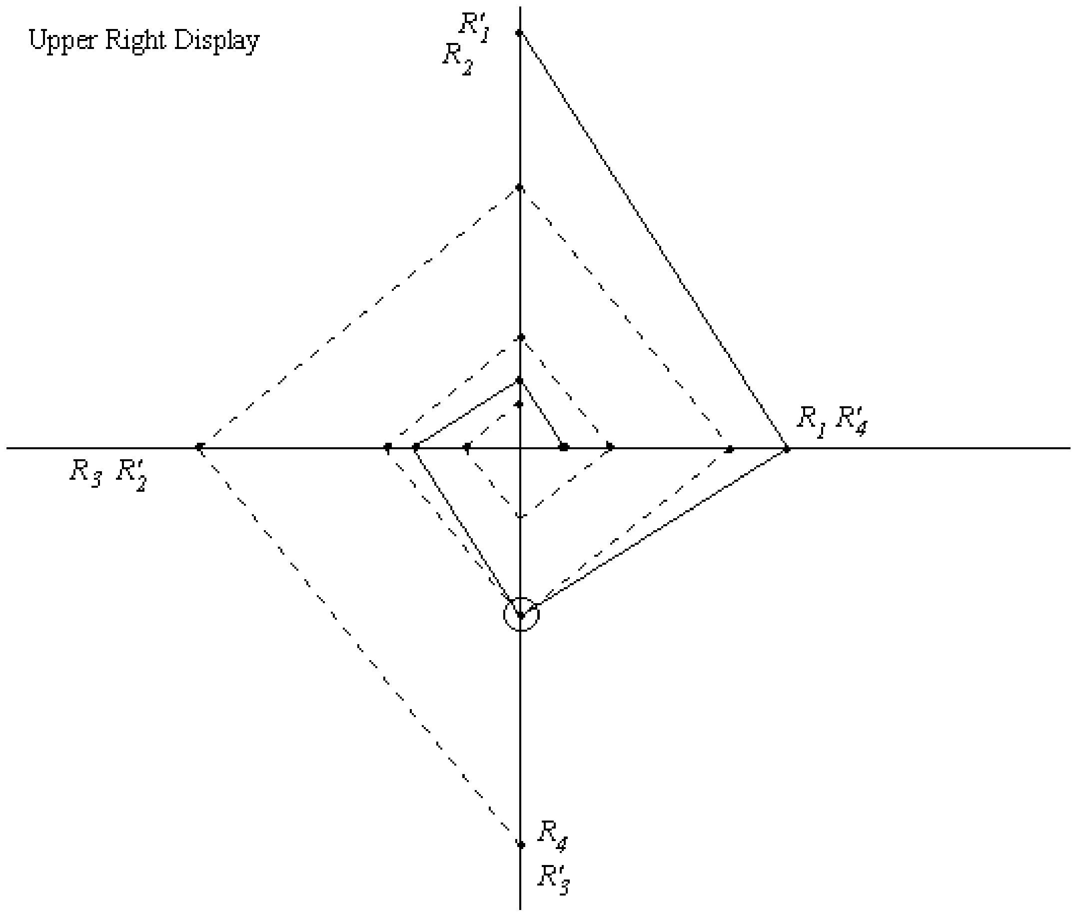
Graphics:

	$4^{n/3}$ (Solid line)				$4^{n/6}$ (Dashed)	
R_1	$4^{0/3}$			R'_1	$4^{0/6}$	
R_2	$4^{1/3}$			R'_2	$4^{1/6}$	
R_3	$4^{2/3}$			R'_3	$4^{2/6}$	
*	R_4	$4^{3/3}$			R'_4	$4^{3/6}$
R_1	$4^{4/3}$			R'_1	$4^{4/6}$	
R_2	$4^{5/3}$			R'_2	$4^{5/6}$	
R_3				*	R'_3	$4^{6/6}$
R_4				R'_4	$4^{7/6}$	
R_1				R'_1	$4^{8/6}$	
R_2				R'_2	$4^{9/6}$	
R_3				R'_3	$4^{10/6}$	
R_4						

Calculator Routines:

$C(4^{n/3})$: ON, 4, y^x, n, /, 3, = ___ $C(4^{n/6})$: ON, 4, y^x, n, /, 6, = ___

Upper Right Display



LOWER LEFT DISPLAY: Two spirals, $5^{n/3}$ and $5^{n/6}$, are displayed such that the initial ruler, R'_1, of the second spiral, $5^{n/6}$, coincides with the third ruler, R_3, of the first spiral, $5^{n/3}$. That is, the system for the second spiral has been rotated $\dfrac{2\pi}{2}$ radians with reference to the system for the first spiral. This display demonstrates $5^{2/3} = 5^{4/6}$.

Graphics:

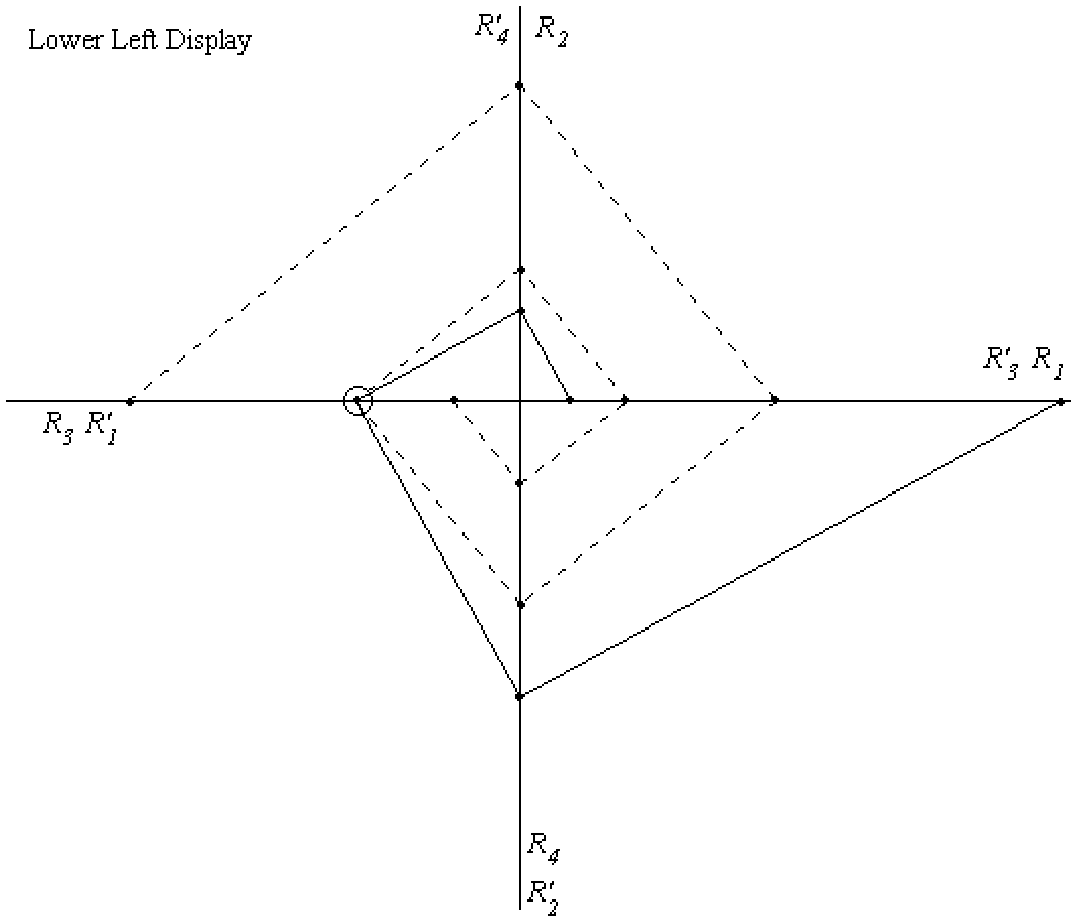
		$5^{n/3}$ (Solid line)			$5^{n/6}$ (Dashed)
	R_1	$5^{0/3}$		R'_1	$5^{0/6}$
	R_2	$5^{1/3}$		R'_2	$5^{1/6}$
*	R_3	$5^{2/3}$		R'_3	$5^{2/6}$
	R_4	$5^{3/3}$		R'_4	$5^{3/6}$
	R_1	$5^{4/3}$	*	R'_1	$5^{4/6}$
	R_2			R'_2	$5^{5/6}$
	R_3			R'_3	$5^{6/6}$
	R_4			R'_4	$5^{7/6}$
	R_1			R'_1	$5^{8/6}$
	R_2				
	R_3				

Calculator Routines:

$C\left(5^{n/3}\right)$: ON, 5, y^x, n, /, 3, = ___

$C\left(5^{n/6}\right)$: ON, 5, y^x, n, /, 6, = ___

Lower Left Display



LOWER RIGHT DISPLAY: Two spirals, $6^{n/4}$ and $6^{n/8}$, are displayed such that the initial ruler, R'_1, of the second spiral, $6^{n/8}$, coincides with the fourth ruler, R_4, of the first spiral, $6^{n/4}$. That is, the system for the second spiral has been rotated $\frac{3\pi}{2}$ radians with reference to the system for the first spiral. This display demonstrates $6^{1/4} = 6^{2/8}$ and $6^{5/4} = 6^{10/8}$.

Graphics:

		$6^{n/4}$ (Solid line)			$6^{n/8}$ (Dashed)
	R_1	$6^{0/4}$		R'_1	$6^{0/8}$
*	R_2	$6^{1/4}$		R'_2	$6^{1/8}$
	R_3	$6^{2/4}$	*	R'_3	$6^{2/8}$
	R_4	$6^{3/4}$		R'_4	$6^{3/8}$
	R_1	$6^{4/4}$		R'_1	$6^{4/8}$
**	R_2	$6^{5/4}$		R'_2	$6^{5/8}$
	R_3			R'_3	$6^{6/8}$
	R_4			R'_4	$6^{7/8}$
	R_1			R'_1	$6^{8/8}$
	R_2			R'_2	$6^{9/8}$
	R_3		**	R'_3	$6^{10/8}$
	R_4				
	R_1				
	R_2				

Calculator Routines:

$C(6^{n/4})$: ON, 6, y^x, n, /, 4, = ___

$C(6^{n/8})$: ON, 6, y^x, n, /, 8, = ___

Lower Right Display