Chaos and Fractals on the TI Graphing Calculator

Linda Sundbye, Ph.D.
Department of Mathematical and Computer Sciences
Metropolitan State College of Denver
Campus Box 38, P.O. Box 173362
Denver, CO 80217
sundbyel@mscd.edu

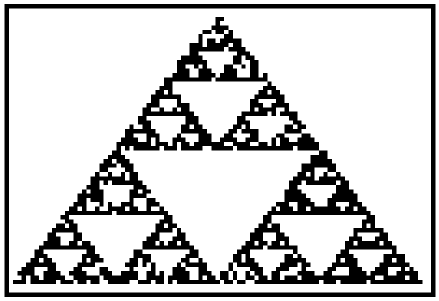
A **Fractal** is a set with fine structure on arbitrarily small scales, with a noninteger dimension, and usually with some degree of self-similarity. Self-similar means that the set looks the same on any scale; That is, smaller pieces reproduce the entire set upon magnification. Fractals occur in many natural objects such as coastlines, mountains, canyons, blood vessels, the lungs, and cauliflower.

1 The Chaos Game

To play this game, we number the vertices of an equilateral triangle 1, 2 and 3. We start with a random initial point in the plane and plot this point. A spinner device is used to randomly pick a number: 1, 2 or 3. From our initial point, we then move half the distance towards that numbered vertex. This point is plotted and becomes our new initial point. We repeat this process a few thousand times.

The picture that is generated through this random process is not random at all. The Sierpinski triangle was introduced by Waclaw Sierpinski in 1916.

The Sierpinski Triangle



2 Iterated Function Systems and Affine Transformations

The Sierpinski triangle can also be generated mathematically from the iterated function system (IFS) (see Barnsley)

$$w_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} .5 & 0 \\ 0 & .5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

$$w_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} .5 & 0 \\ 0 & .5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .25 \\ .50 \end{pmatrix} \tag{2}$$

$$w_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} .5 & 0 \\ 0 & .5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .50 \\ 0 \end{pmatrix} \tag{3}$$

The function w_i maps the point (x, y) to a new coordinate. The algorithm is:

Randomly pick initial condition (x, y).
For $K = 1, 3000$, randomly pick $p \in [0, 1]$
If $p \leq 1/3$, then map the point (x, y) by w_1
If $1/3 < p \leq 2/3$, then map the point (x, y) by w_2
If $2/3 < p$, then map the point (x, y) by w_3
Plot point. Repeat with this point as initial condition.

The mapping $w(x, y)$ is called an *affine transformation*. This type of transformation stretches (or shrinks) and translates.

In one dimension, the linear equation $w(x) = ax + b$ is an affine transformation. In two dimensions, the mapping

$$w\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

is an affine transformation.

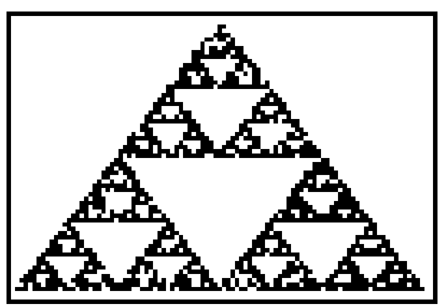
Codes for several iterated function systems are given below. The quantity p represents the probability of calling the function w_i. Thus, for rotationally symmetric fractals, the functions w_i have equal probabilities.

New fractal images may easily be generated by students by changing the parameter values, probabilities, and by changing the number of equations. (See Barnsley or Scheinerman)

IFS code for the Sierpinski triangle

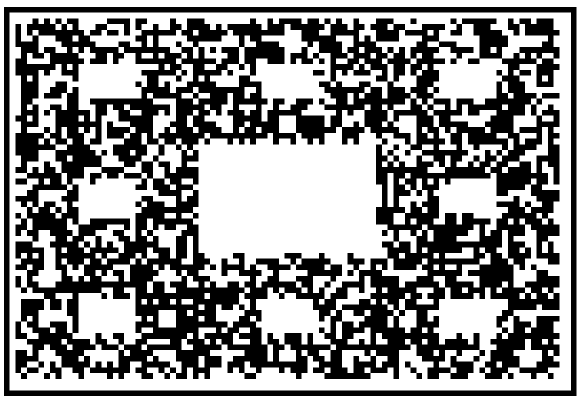
w	a	b	c	d	e	f	p
1	1/2	0	0	1/2	0	0	1/3
2	1/2	0	0	1/2	1/4	1/2	1/3
3	1/2	0	0	1/2	1/2	0	1/3

The following pictures were generated on the TI–83 graphing calculator.



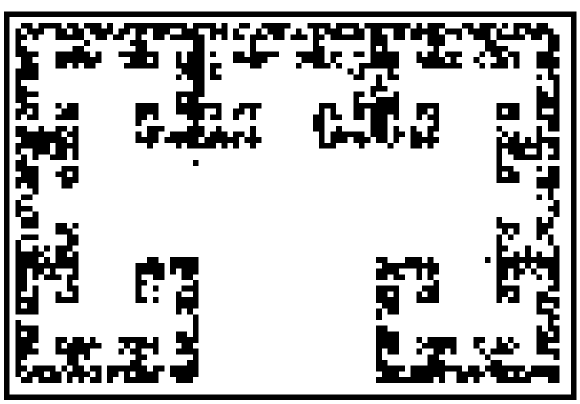
IFS code for the Sierpinski carpet

+w	a	b	c	d	e	f	p
+1	1/3	0	0	1/3	0	0	1/8
+2	1/3	0	0	1/3	1/3	0	1/8
+3	1/3	0	0	1/3	2/3	0	1/8
+4	1/3	0	0	1/3	0	1/3	1/8
+5	1/3	0	0	1/3	2/3	1/3	1/8
+6	1/3	0	0	1/3	0	2/3	1/8
+7	1/3	0	0	1/3	1/3	2/3	1/8
+8	1/3	0	0	1/3	2/3	2/3	1/8



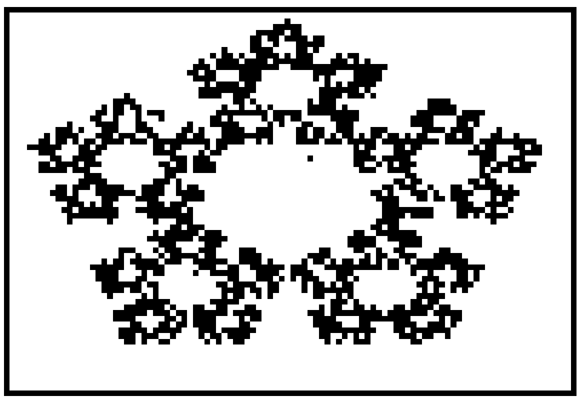
IFS code for the Cantor maze

+w	a	b	c	d	e	f	p
+1	1/3	0	0	1/3	1/3	2/3	1/7
+2	0	1/3	1	0	2/3	0	3/7
+3	0	–1/3	1	0	1/3	0	3/7



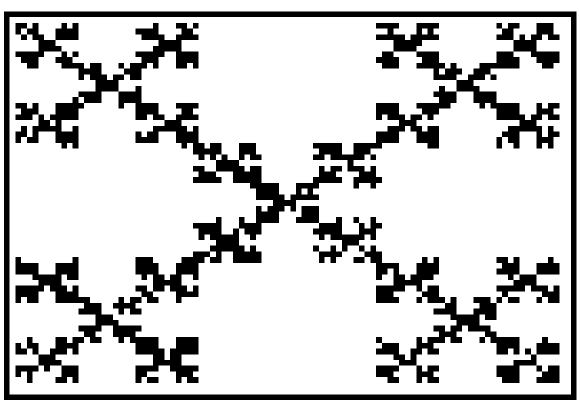
IFS code for a 5-sided crystal

+w	a	b	c	d	e	f	p
+1	0.382	0	0	0.382	0.3072	0.6190	1/5
+2	0.382	0	0	0.382	0.6033	0.4044	1/5
+3	0.382	0	0	0.382	0.0139	0.4044	1/5
+4	0.382	0	0	0.382	0.1253	0.0595	1/5
+5	0.382	0	0	0.382	0.4920	0.0595	1/5



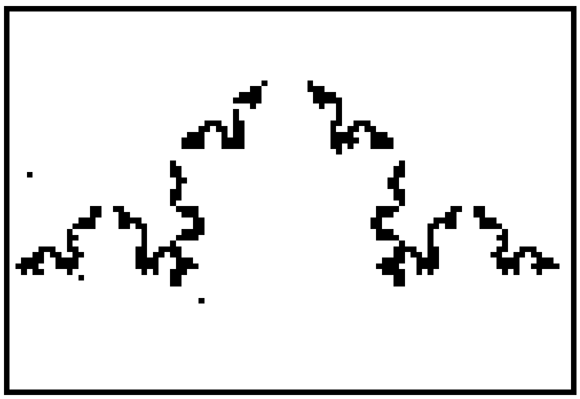
IFS code for a fractal

+w	a	b	c	d	e	f	p
+1	1/3	0	0	1/3	0	0	1/5
+2	1/3	0	0	1/3	2/3	0	1/5
+3	1/3	0	0	1/3	0	2/3	1/5
+4	1/3	0	0	1/3	2/3	2/3	1/5
+5	1/3	0	0	1/3	1/3	1/3	1/5



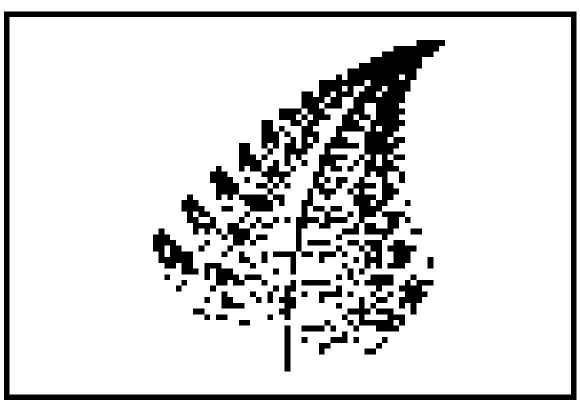
IFS code for the Koch curve

+w	a	b	c	d	e	f	p
+1	1/3	0	0	1/3	0	0	1/4
+2	1/6	$\frac{-1}{\sqrt{12}}$	$\frac{1}{\sqrt{12}}$	1/6	1/3	0	1/4
+3	1/6	$\frac{1}{\sqrt{12}}$	$\frac{-1}{\sqrt{12}}$	1/6	1/2	$\frac{1}{\sqrt{12}}$	1/4
+4	1/3	0	0	1/3	2/3	0	1/4



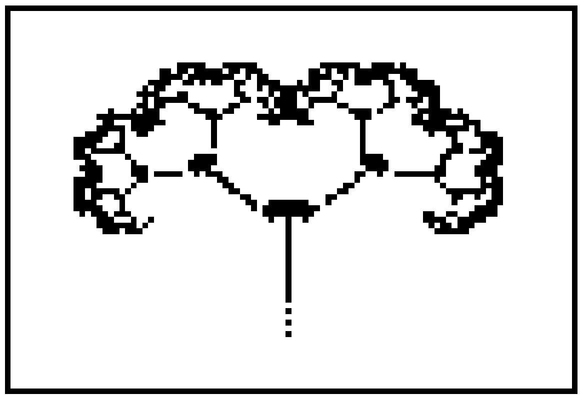
IFS code for a fern

w	a	b	c	d	e	f	p
1	0	0	0	0.16	0	0	0.01
2	0.85	0.04	–0.04	0.85	0	1.6	0.85
3	0.2	–0.26	0.23	0.22	0	1.6	0.07
4	–0.15	0.28	0.26	0.24	0	0.44	0.07



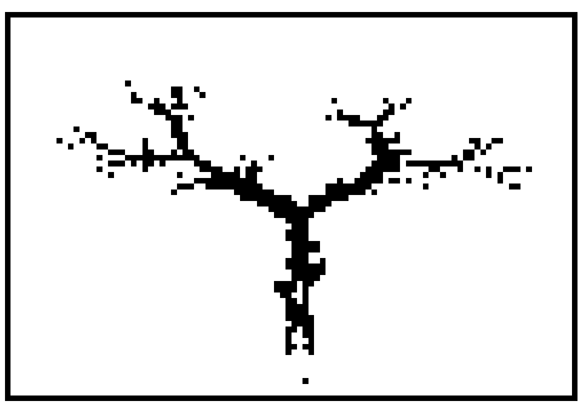
IFS code for a tree

+w	a	b	c	d	e	f	p
+1	0	0	0	0.5	0	0	0.05
+2	0.42	–0.42	0.42	0.42	0	0.2	0.40
+3	0.42	0.42	–0.42	0.42	0	0.2	0.40
+4	0.1	0	0	0.1	0	0.2	0.15



IFS code for another tree

+w	a	b	c	d	e	f	p
+1	0.195	–0.488	0.344	0.443	0.4431	0.2452	0.1699
+2	0.462	0.414	–0.252	0.361	0.2511	0.5692	0.1811
+3	–0.058	–0.070	0.453	–0.111	0.5976	0.0969	0.2161
+4	–0.035	0.070	–0.469	–0.022	0.4884	0.5069	0.2198
+5	–0.637	0	0	0.501	0.8562	0.2513	0.2131



3 The Dimension of Self-Similar Fractals

A power law relationship exists between the number of pieces N and the reduction factor r:

$$N = \frac{1}{r^D}$$

Solving this equation for D gives the **similarity dimension:**

$$D_s = \frac{\ln N}{\ln(1/r)}$$

For the Koch curve, the first iteration K_1 has $N = 4$ pieces and a reduction factor of $r = 1/3$. Therefore, $D_s = \ln 4/\ln 3 = 1.26186.....$

Similarly, K_2 has $N = 16$ pieces and a reduction factor of $r = 1/9$ from the original. Therefore, $D_s = \ln 4/\ln 3$.

Cantor's middle third set has $N = 2$ pieces and a reduction factor of $r = 1/3$ at all levels. Therefore, $D_s = \ln 2/\ln 3 = 0.63029...$

The Sierpinski triangle has $N = 3$ pieces and a reduction factor of $r = 1/2$. And $D_s = \ln 3/\ln 2 = 1.58496....$

The Sierpinski carpet has $N = 8$ pieces and a reduction factor of $r = 1/3$. And $D_s = \ln 8/\ln 3 = 1.89279....$

For fractals that are not self-similar, we need yet a different definition of dimension. There are many other definitions of dimension such as the fractal dimension, the Hausdorff dimension and the box dimension, which is a special case of the fractal dimension. (See Barnsley, or Scheinerman)

4 References

[**1**] Alligood, K. T. Sauer and J. Yorke, *Chaos, An Introduction to Dynamical Systems*, Springer-Verlag, New York, (1997).

[**2**] Baker, G. and J. Gollub, *Chaotic Dynamics, An Introduction, Second Edition*, Cambridge University Press, Cambridge, England, (1996).

[**3**] M. Barnsley, *Fractals Everywhere, Second Edition,* Academic Press, Boston, (1993).

[**4**] J. Gleick, *Chaos, Making a New Science*, Penguin, New York, (1987).

[**5**] Mandelbrot, B., *The Fractal Geometry of Nature*, Freeman, San Francisco, (1982).

[**6**] Peitgen, H.O., H. Jurgens, and D. Saupe, *Fractals for the Classroom, Part One, Introduction to Fractals and Chaos*, Springer Verlag, NCTM, (1992).

[**7**] Peitgen, H.O., H. Jurgens, and D. Saupe, *Fractals for the Classroom, Part Two, Complex Systems and Mandelbrot Set*, Springer Verlag, NCTM, (1992).

[**8**] Scheinerman, E., *Invitation to Dynamical Systems*, Prentice Hall, New Jersey, (1996).