Cubic Equations-Their Presence, Importance, and Applications, in the Age of Technology

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ABSTRACT: I in this article we will attempt, with the help of modern mathematical technology, to revive an interest and research among teachers and students in cubic and higher order equations. The article, written in attractive *Mathematica's* notebook format, basically comprises two themes: It uses *Mathematica* 3.0 to discover simple and natural methods, algebra and calculus based methods, for solving exactly a cubic equation. Once the idea is discover, then the computation can be easily carried out by a hand, by anyone with standard algebra or calculus skills. In the second part we will show variety of different problems whose solution requires solving a non-trivial cubic equation. In that part we will take advantage of *Mathematica* to do the computation.

The problem that infected me with such virulence was actually of little significance, and even lesser consequence. It concerned solving cubic equations, and the answer had been known since Cardano published it in 1545. What I did not know was how to drive it.

The sages who had designed the mathematics curricula for secondary schools had stopped at solving quadratic equations...

Mark Kac

How I Became a Mathematician

American Scientist, Sep/Oct 1984

They were very upset when I said the development of the greatest importance to mathematics in Europe was the discovery by Tartaglia that you can solve a cubic equation: although it is of little use in itself, the discovery must have been psychologically wonderful. It therefore helped in the Renaissance, which was freeing man from the intimidation of ancients. What the Greeks are learning in the school is to be intimidated into thinking they have fallen so far bellow their ancestors.

Richard Feynman, physicist,

comments about educational attitudes he had met during his visit to Greece in 1980.

What Do you Care What Other People Think?, Unwin, 1988

Discovering Solution of Cubic Equation

■ Taking Advantage of Traditional Notation

We will restrict ourselves, as usually in the theory of cubic equations, to the equations of the form $x^3+px+q=0$. For any general cubic equation $a x^3 + b x^2 + c x + d = 0$ can be easily converted to the previous one by the means of the transformation $x = y - \frac{b}{}$.

For example: Consider the equation $5 x^3 + 4 x^2 - 2 x + 3 = 0$, and apply substitution $x = y - \frac{4}{3 * 5}$

$$5 x^3 + 4 x^2 - 2 x + 3 \; /. \; x \to y - \frac{4}{3 * 5}$$

$$3 - 2 \left(-\left(\frac{4}{15}\right) + y\right) + 4 \left(-\left(\frac{4}{15}\right) + y\right)^2 + 5 \left(-\left(\frac{4}{15}\right) + y\right)^3$$

```
Simplify[%]
```

$$\frac{2513}{675} - \frac{46 y}{15} + 5 y^3$$

Solve a few equations by using *Mathematica*'s command Solve, and make an observation about their solution structure. To make this observation easier, use the traditional output format. For example:

```
Solve[2513/675 - 46 y/15 + 5 y^3 == 0, y]
```

$$\left\{\left\{y\to -\frac{1}{15}\sqrt[3]{\frac{1}{2}\left(2513-15\sqrt{26337}\right)}+-\frac{46}{15}\sqrt[3]{\frac{2}{2513-15\sqrt{26337}}}\right\},\right.$$
$$\left\{y\to \frac{1}{30}\left(1-i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(2513-15\sqrt{26337}\right)}+\frac{23}{15}\left(i\sqrt{3}+1\right)\sqrt[3]{\frac{2}{2513-15\sqrt{26337}}}\right\},$$
$$\left.\left\{y\to \frac{1}{30}\left(i\sqrt{3}+1\right)\sqrt[3]{\frac{1}{2}\left(2513-15\sqrt{26337}\right)}+\frac{23}{15}\left(1-i\sqrt{3}\right)\sqrt[3]{\frac{2}{2513-15\sqrt{26337}}}\right\}\right\}$$

```
Solve[x^3 - 3 x - 4 == 0, x]
```
(1)

$$\left\{\left\{x\to\sqrt[3]{\sqrt{3}+2}+\frac{1}{3}\sqrt[3]{54-27\sqrt{3}}\right\},\left\{x\to-\frac{1}{2}\left(\left(1-i\sqrt{3}\right)\sqrt[3]{\sqrt{3}+2}\right)-\frac{1}{6}\sqrt[3]{54-27\sqrt{3}}\left(i\sqrt{3}+1\right)\right\},\right.$$
$$\left.\left\{x\to-\frac{1}{2}\left(\left(i\sqrt{3}+1\right)\sqrt[3]{\sqrt{3}+2}\right)-\frac{1}{6}\sqrt[3]{54-27\sqrt{3}}\left(1-i\sqrt{3}\right)\right\}\right\}$$

We can observe that a solution, especially the real one, is of a form

$$x = \sqrt[3]{u} + \sqrt[3]{v}.$$

We will use this fact, to find a simple and natural way for solving a cubic equation. We will illustrate this idea by solving the equation (1). The equation (1) we will rewrite as

$$x = \sqrt[3]{3x+4},$$

Then set up, as observed,

$$x = \sqrt[3]{u} + \sqrt[3]{v}$$

and

$$\sqrt[3]{3x+4} = \sqrt[3]{u} + \sqrt[3]{v}.$$

By cubing both sides, we get

$$x + 4 = u + v + 3\sqrt[3]{uv}\left(\sqrt[3]{u} + \sqrt[3]{v}\right),$$

but $3x+4$, by using again the fact that $x = \sqrt[3]{u} + \sqrt[3]{v}$, can be expressed as

$$3x + 4 = 4 + 3\left(\sqrt[3]{u} + \sqrt[3]{v}\right).$$

Thus,

$$4 + 3\left(\sqrt[3]{u} + \sqrt[3]{v}\right) = u + v + 3\sqrt[3]{uv}\left(\sqrt[3]{u} + \sqrt[3]{v}\right),$$

and so, the solution of the cubic equation is reduced to a solution of this system of equations

$$u + v = 4, \text{ and } 3\sqrt[3]{uv} = 1,$$

that is

$u+v=4,\ \text{and}\ u\,v=1$

The last system of equations can be reduced to the quadratic equation

$u^2-4u+1=0$

which has two real solutions, however the expression $\sqrt[3]{u}+\sqrt[3]{v}$ won't be affected by the choice (verify this). Thus we can take as $u=2-\sqrt{3}$, and $v=2+\sqrt{3}$, and hence our solution to the equation (1) is

$x=\sqrt[3]{2-\sqrt{3}}+\sqrt[3]{2+\sqrt{3}}$,

which can be verified that is the same as the one obtained by *Mathematica.*

In this example we will consider the equation with all three real roots.
$t^3+6\,t\text{^}2+6\,t-3=0.$

Complete cube

$t+2)^3-6\,(t+2)+1=0,$

and rewrite it as

$(t+2)^3=6\,(t+2)-1.$

Set up

$t+2=\sqrt[3]{u}+\sqrt[3]{v}$,

Then,

$6\,(t+2)-1=u+v+3\sqrt[3]{u\,v}\left(\sqrt[3]{u}+\sqrt[3]{v}\right),$

$6\,(t+2)-1=-1+6\left(\sqrt[3]{u}+\sqrt[3]{v}\right).$

Solve the system of equations

$u+v=-1,\ 3\sqrt[3]{u\,v}=6$

Its solution is

$\left\{\left\{u\to\frac{1}{2}\left(-i\sqrt{31}-1\right),\ v\to\frac{1}{2}\left(i\sqrt{31}-1\right)\right\},\ \left\{u\to\frac{1}{2}\left(i\sqrt{31}-1\right),\ v\to\frac{1}{2}\left(-i\sqrt{31}-1\right)\right\}\right\}$

Represent u and v in trigonometric form.

$u=2\sqrt{2}\left(\cos\left(\tan^{-1}\left(\sqrt{31}\right)-\pi\right)+i\sin\left(\tan^{-1}\left(\sqrt{31}\right)-\pi\right)\right)$

$v=2\sqrt{\ }\left(\cos\left(\pi-\tan^{-1}\left(\sqrt{31}\right)\right)+i\cos\sin\left(\pi-\tan^{-1}\left(\sqrt{31}\right)\right)\right)$

Compute the first set of cube roots of u and v.

$\sqrt[3]{u}=\sqrt[3]{2\sqrt{2}}\left(\cos\left(\tan^{-1}\left(\frac{\sqrt{31})-\pi}{3}\right)+i\sin\left(\tan^{-1}\left(\frac{\sqrt{31})-\pi}{3}\right)\right)\right)$

$\sqrt[3]{v}=\sqrt[3]{2\sqrt{2}}\left(\cos\left(\frac{\pi-\tan^{-1}\left(\sqrt{31}\right)}{3}\right)+i\sin\left(\frac{\pi-\tan^{-1}\left(\sqrt{31}\right)}{3}\right)\right)$

Since the above complex numbers are conjugate to each other, their sum is real. Thus, the solution of our equation is

$\mathbf{t+2} = 2\sqrt[3]{2\sqrt{2}}\left(\cos\left(\frac{\pi-\tan^{-1}(\sqrt{31})}{3}\right)\right)$, or $\mathbf{t} = -2+2\sqrt[3]{2\sqrt{2}}\left(\ \right)$

The other two roots of the cubic equation, we will get by considering the other two cube roots of u and v.

$\mathbf{t} = -2+2\sqrt[3]{2\sqrt{2}}\left(\cos\left(\frac{\pi-\tan^{-1}(\sqrt{31})+2\pi}{3}\right)\right)$, and

$\mathbf{t} = -2+2\sqrt[3]{2\sqrt{2}}\left(\cos\left(\frac{\pi-\tan^{-1}(\sqrt{31})+4\pi}{3}\right)\right)$.

■ Solving Cubic Equation in Calculus

We will demonstrate now a completely different approach to solution of a cubic $x^3+p\,x+q=0$, by using calculus tools, differentiation, integration, saparation of variables. According to [1], this method was already published by John Landen in 1775. The q in the cubic equation we will treat as a function of x. Thus

$q = -x^3 - px = -x\,(x^2+p)$, $q' = -3\,x^2 - p$, $q'' = -6\,x$, and

$q[0] = 0$, $q'[0] = -p$, $q''[0] = 0$.

We can observe that

$q'' = \frac{-18\,q}{q'-2\,p}$,

from which we get that

$q''\,q' - 2\,p\,q = -18\,q$,

multiplying by q',

$q''\,(q')^2 - 2\,p\,q\,q' = -18\,q\,q'$,

integrating both sides

$(q')^3 - 3\,p\,(q')^2 = -27\,(q)^2 + c$.

By using the initial conditions, we will find that

$c = -4\,p^3$.

Thus we have that $(q')^3 - 3\,p\,(q')^2 = -27\,(q)^2 - 4\,p^3$, that is

$(q')^2\,(q' - 3\,p) = -27\,(q)^2 - 4\,p^3$.

By substituting explicit formula for q' in $q' - 3\,p$, on the left side, we get that

$(q')^2\,(3\,x^2 + 4\,p) = 27\,(q)^2 + 4\,p^3$.

It can be observed now that the above differential equation can be solved by separating variables.

$$\frac{dx}{\sqrt{3\,x^2+4\,p}} = -\frac{dq}{\sqrt{27\,q^2+4\,p^3}}$$

If $p > 0$, the integration will yield

$$\frac{1}{\sqrt{3}}\,\mathrm{ArcSinh}\left(\sqrt{\frac{3}{4\,p}}\;x\right) = -\frac{1}{\sqrt{27}}\,\mathrm{ArcSinh}\left(\sqrt{\frac{27}{4\,p^3}}\;q\right),$$

from which

$$x = -\sqrt{\frac{4p}{3}}\ \mathrm{Sinh}\left(\frac{1}{3}\ \mathrm{ArcSinh}\left(\sqrt{\frac{27}{4p^3}}\ q\right)\right).$$

If $p < 0$, the integration will yield , in the above formula ArcSin in place of ArcSinh.

$$x = \sqrt{-\frac{1}{3}(4p)}\ \mathrm{Sin}\left(\frac{1}{3}\left(\mathrm{ArcSin}\left(\left[\sqrt{-\frac{27}{4p^3}}\ q\right)\right)\right)$$

Since, in the above formula, ArcSin is a multivalued function, by selecting an appropriate branch, that is *n* in the formula below, we can find all the roots.

Namely, $x = \pm\sqrt{-\frac{1}{3}(4p)}\ \mathrm{Sin}\left(\frac{1}{3}\left(n\pi + \mathrm{ArcSin}\left(\left[\sqrt{-\frac{27}{4p^3}}\ q\right)\right)\right)$

We would like to write a program that will compute all real roots for a real cubic equation by using the formulas with Sin and Sinh.

First, however, we will need to derive a well -known formula for discriminant. The discrimminant is

is defined as $\mathrm{Discr} = -\frac{1}{108}(x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2$

If $D > 0$, then the cubic equation has only one real root. If $D < 0$, then the cubic equation has three distinct roots. If Discr=0, then the cubic equation has all three roots real, but two of them are repeated.

The constant $\frac{-1}{108}$ is introduced in order to obtain convenient easily remembered final form.

```
s = x /. Solve[x^3 + p x + q == 0, x];
Discr[p_, q_] = FullSimplify[
  -1/108 (x1 - x2)^2 (x1 - x3)^2 (x2 - x3)^2 /. {x1 -> s[[1]], x2 -> s[[2]], x3 -> s[[3]]}]
```

```
 3    2
p    q
-- + --
27   4
```

Now we are in position to define, in *Mathematica*'s syntax, the function **realroots** that will compute the real roots of a cubic equation $x^3 + p\,x + q = 0$, p and q≠0 , and two auxiliary function **rootp** which will compute the only real root in the case $p > 0$ since discriminent is positive, and function **rootn** which will compote the real roots in the case $p < 0$.

```
rootp[p_, q_] := N[-Sqrt[4 p/3] Sinh[1/3 ArcSinh[Sqrt[27/(4 p^3)] q]]];
rootn[p_, q_, n_] := (-1)^n N[Sqrt[-1/3 (4 p)] Sin[1/3 (π n + ArcSin[Sqrt[-27/(4 p^3)] q])]];
```

$$\texttt{rootp[p\_, q\_] := N}\left[-\sqrt{\frac{4\,p}{3}}\ \texttt{Sinh}\left[\frac{1}{3}\ \texttt{ArcSinh}\left[\sqrt{\frac{27}{4\,p^3}}\ q\right]\right]\right];$$

$$\texttt{rootn[p\_, q\_, n\_] := }(-1)^n\ \texttt{N}\left[\sqrt{-\frac{1}{3}\,(4\,p)}\ \texttt{Sin}\left[\frac{1}{3}\left(\pi\,n + \texttt{ArcSin}\left[\sqrt{-\frac{27}{4\,p^3}}\ q\right]\right)\right]\right];$$

```
General::spell1:
   Possible  spelling  error:  new symbol name "rootn"
     is similar  to existing  symbol  "rootp".
```

```
realroots[x_^3 + p_ x_ + q_] :=
 Which[
  p > 0, rootp[p, q],
  p < 0 && q < 0 && Discr[p, q] > 0, rootn[ p, q, 2],
  p < 0 && q > 0 && Discr[p, q] > 0, rootn[ p, q, 1],
   p < 0 && Discr[p, q] <= 0, {rootn[p, q, 0], rootn[p, q, 1], rootn[p, q, 2]},
p == 0, -∛q ] // Chop
```

Now we will examine how **realroots** function works in practice, pointing out possible problems with their solutions.

Example: One real root, p > 0, $x^3 + 2x + 4 == 0$

```
realroots[x^3 + 2 x + 4]
```

```
-1.17951
```

```
NSolve[x^3 + 2 x + 4 == 0, x]
```

```
{{x -> -1.17951}, {x -> 0.589755 - 1.74454 I},
  {x -> 0.589755 + 1.74454 I}}
```

Comments: **realroots** finds correctly the only one solution

Example: One real root, p < 0, $x^3 - 2x + 3 == 0$.

```
realroots[x^3 - 2 x + 3]
```

```
-1.89329
```

```
NSolve[x^3 - 2 x + 3 == 0, x]
```

```
{{x -> -1.89329}, {x -> 0.946645 - 0.829704 I},
  {x -> 0.946645 + 0.829704 I}}
```

Comments: **realroots** finds correctly the only one solution

Example: Three real roots, p < 0, $x^3 - 3x - 1 == 0$

```
realroots[x^3 - 3 x - 1]
```

```
{-0.347296, -1.53209, 1.87939}
```

```
NSolve[x^3 - 3 x - 1 == 0, x]
```

```
{{x -> -1.53209}, {x -> -0.347296}, {x -> 1.87939}}
```

Comments: **realroots** finds correctly all the solutions

Example: Repeated roots, $x^3 - 3\sqrt[3]{0.25}\,x - 1 == 0$

```
realroots[x^3 - 3 (0.25)^(1/3) x - 1]
```

```
1.5874
```

```
NSolve[x^3 - 3 (0.25)^(1/3) x - 1 == 0, x]
```

```
{{x -> -0.793701}, {x -> -0.793701}, {x -> 1.5874}}
```

Comments: **realroots** failed to find all the roots. Reason, the **Discr** funtion was not evaluated to zero because the coefficients were entered numerically.

```
realroots[x^3 - 3 (1/4)^(1/3) x - 1]
```
$\texttt{realroots}\left[x^3 - 3\sqrt[3]{\frac{1}{4}}\,x - 1\right]$

```
{-0.793701, -0.793701, 1.5874}
```

■ Solution of Cubic Equation by Mark Kac

In the book *Mathematics and Logic,* Mark Kac provides yet another method to solve a cubic equation. A method discovered by himself in his youth. He made an observation, using alternating groups and symmetric polynomials, that the expressions $f(x,\ y,\ z) := (z\,w^2 + y\,w + x)^3$ and $g(x,\ y,\ z) := (y\,w^2 + z\,w + x)^3$, where w is the cubic root of unity and x,y,z are the roots of a cubic equation, can be solely expressed by the coefficients of a cubic equations. In this section we will assume that a cubic equation is of the form $x^3 + px + q = 0$. Then by using exact formulas for the roots, obtained by applying **Solve** command , we compute f and g at these roots and simplify the result by using the command **FullSimplify.**

```
Clear[w, f, g, a, b]
```

$w = -\frac{1}{2} + \frac{\sqrt{3}\,i}{2}$; **Expand**[$w^3$]

```
1
```

$f(\text{x\_},\ \text{y\_},\ \text{z\_}) := (z\,w^2 + y\,w + x)^3$; **roots** = x /. **Solve**($x^3 + p\,x + q == 0,\ x$);
FullSimplify(f @@ **roots**)

```
General::spell:
   Possible  spelling  error:  new  symbol  name  "roots"
     is  similar  to  existing  symbols  {rootn,  rootp,  Roots}.
```

```
                             3          2
-3 (9 q + Sqrt[3] Sqrt[4 p  + 27 q ])
-------------------------------------
                  2
```

Since *Mathematica* renders a cube root of a negative real number as a complex number, we would like it to be a real number, we defined our own cube root function.

$\texttt{cuberoot[x\_] := If}\left[\texttt{Re[x]} < 0,\ w\sqrt[3]{x},\ \sqrt[3]{x}\right]$

```
a[p_, q_] := cuberoot[-3/2 (9 q + Sqrt[3] Sqrt[4 p^3 + 27 q^2])]
```

$g(x\_, y\_, z\_) := (z\,w + y\,w^2 + x)^3$; roots = x /. Solve($x^3 + p\,x + q == 0$, x);
FullSimplify(g @@ roots)

```
                       3        2
-27 q + Sqrt[108 p  + 729 q ]
---------------------------------
               2
```

```
b[p_, q_] := cuberoot[1/2 (Sqrt[108 p^3 + 729 q^2] - 27 q)]
```

By applying the fact that $x + y + z = 0$, we can set up a system of linear equations.

```
kacway[p_, q_] :=
 NSolve[{x + y + z == 0, z w^2 + y w + x == a[p, q], y w^2 + z w + x == b[p, q]}, {x, y, z}]
```

```
kacway[-2, 4] // Chop
```

```
{{x -> -2., y -> 1. + 1. I, z -> 1. - 1. I}}
```

Mathematica's solution.

NSolve($x^3 - 2\,x + 4 == 0$)

```
{{x -> -2.}, {x -> 1. - 1. I}, {x -> 1. + 1. I}}
```

Unfortunately, **kacway** does not always produces a correct solution. This problem is caused by multiple-valuedroot functions in complex domain. One way to solve this problem, probably not the best, is to multiply the right side of the system of equation by w or w^2. In other words, consider different branch For example:

```
NSolve[x^3 - 4 x + 1 == 0]
```

```
{{x -> -2.11491}, {x -> 0.254102}, {x -> 1.86081}}
```

But, **kacway** will do incorrectly.

```
kacway[-4, 1] // Chop
```

```
{{x -> -0.930403 + 1.61151 I, y -> 1.05745 - 1.83156 I,
   z -> -0.127051 + 0.220059 I}}
```

To correct this problem, consider

```
kacway2[p_, q_] :=
 NSolve[{x + y + z == 0, z w^2 + y w + x == w^2 * a[p, q], y w^2 + z w + x == w^2 * b[p, q]},
  {x, y, z}]
```

```
kacway2[-4, 1] // Chop
```

```
{{x -> 1.86081, y -> -2.11491, z -> 0.254102}}
```

Problems with Cubic Eqauation

- **Making Box**
- **Height of Water in Spherical Tank**
- **The Smallest Distance from a Parabola**
- **Pumping Water out of Tank**
- **Equation of State for Real Gases**
- **Electrical Resistance**
- **Finding Interest Rate**
- **Break-Even Points in Economics**
- **Von Neumann's Model of an Expanding Economy**

- **Trisecting the Angle**
- **After the proof of Feramat's Last Theorem ,Andrew Wiles, Open Problem for the Next Century**

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