COMPUTER LABORATORY IMPLEMENTATION ISSUES
AT A SMALL LIBERAL ARTS COLLEGE

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Abstract:
Lycoming College is a small, private, liberal arts college in central Pennsylvania. The Department of Mathematical Sciences has recently added weekly computer labs using *Derive* to PreCalculus, Calculus I, and Calculus II. This presentation will discuss various issues that have arisen in the implementation of these labs.

Overview:
Lycoming College has a student body of approximately 1500 students. The Department of Mathematical Sciences consists of seven full-time faculty and several part-time adjuncts. Each semester the department offers, among other courses, one section each of PreCalculus, Calculus I, and Calculus II. Until 1994 these courses were taught in the traditional lecture format. In 1992 the department decided to include computer laboratories to these courses. A committee of three faculty members was established to evaluate Computer Algebra Systems (CAS's). The author was chosen as chair of this committee. Once *Derive* was chosen and purchased, I was put in charge of implementing weekly laboratories in the three courses. Weekly labs were added to PreCalculus starting in the Fall semester of 1993, and in Calculus I and II starting in the Fall semester of 1994. The past academic year was used to "fine-tune" these labs.

Evaluation of CAS's:
The department initially considered the use of graphing calculators. However, it was felt that the additional cost would be a problem for many of our students. With this in mind we decided to investigate different Computer Algebra Systems for use on the college computer network. Our first step was to see what had been accomplished at other schools. I attended the Fifth Annual International Conference on Technology in Collegiate Mathematics in 1992, where I attended several extremely useful workshops using *Derive*, *Mathematica*, and *Maple V*. I also attended various short presentations on CAS applications at that conference. The committee read software reviews found in *PC Magazine* and other sources. We then contacted various companies for evaluation copies (some of which we had to purchase).

After "hands-on" experience with several different software packages we chose *Derive* for our labs. While not as powerful as some of the other CAS's, *Derive*'s menu system made learning the package much easier and faster than most of the other systems. Furthermore, its ability to factor polynomials in a notation understandable to a first-year college student made *Derive* a better choice for PreCalculus. The department hopes to add *Maple V* at some future point for use in upper-level mathematics courses.

Description of Lab Purpose:

We intended these weekly labs as an additional way for students to learn the material. That is, the labs would add to, not replace, the regular lectures. As much as possible, the labs should be exploratory, leading the students to discover facts "by themselves." During labs, students would work in small groups of two or three. Each lab would include a writing component. The concentration would be on the mathematics involved, not on the software being used. Effort should also be taken to demonstrate both the strengths and weaknesses of Computer Algebra Systems.

Writing Labs:

While there were many fine laboratory manuals available from various publishers (with even more available now), at that time none seem designed for weekly labs. For example, one manual had a total of 19 labs covering all of Calculus I, II, and III. With a 14-week semester we wanted a minimum of 14 labs for each course. Also, the lab manuals are to some degree tied to particular textbooks and the ones we were currently using did not have any lab manuals in *Derive*. In order to get the best "fit" with our students, using our texts, it was decided that one of us would write the labs for these courses.

During the summer of 1993 I prepared labs for PreCalculus. I then taught the course using these labs in both the fall and spring semesters of that year. Minor adjustments were made during the spring semesters, and more major changes were made over the summer of 1994. A lab from this course is included as an example at the end of this paper. During that same summer, along with another faculty member, I prepared labs for Calculus I and II. In the fall I taught Calculus I and the other faculty member taught Calculus II, both using the appropriate labs from this collaboration. A lab from these courses is also included at the end of this paper. During the spring semester, while on sabbatical, I continued revising the labs for all three courses while others taught the courses using the labs I prepared. An unfortunate result of switching textbooks in Calculus that semester was the need to reorganize the order of the Calculus labs. The finished labs have now been used for one full year.

Reactions:

Student reaction to the labs varied greatly. Most negative comments were directed at the addition of two contact hours per week. (This complaint was also made by some of the faculty.) Initially, students questioned the value of what they saw as a repetition of material from the lectures. However, as time went on, most students came to appreciate the new approach inherent in the more graphical methods employed in the labs. Surprisingly, the students who resisted the longest were those from near the "top" of the class. The best students were already used to looking at material in more then one way and approached the lab with an almost playful attitude. Some of the students just "below" this level seemed content with their level of understanding, and were annoyed at being asked to approach the material in a different way.

The laboratory format seemed to have the greatest impact on students who were struggling in the lecture format. The most dazzling example of this was one student who was extremely unmotivated in class. On lecture days she would arrive late, seldom turned in

homework, and always seemed bored with the class and subject matter. On lab days she arrived early, always completed each lab, and frequently stayed after the lab to continue "playing" with *Derive*. Clearly, for this student the graphical approach presented a much more understandable way to look at mathematics.

As an educator I was particular impressed with the student involvement during the laboratories. Working in small groups of two or three makes it possible for even shy students to take part. In the beginning of the courses, students sometimes spend too much time concentrating on the "keystrokes" instead of the mathematics. However, this problem decreases as the students get more familiar with the software package. Freeing students from the "drudgery" of calculations really does lead to more discussion on the "how" and "why" of the mathematics involved.

As author of the labs, I was extremely pleased with some of the labs and slightly disappointed with some of the others. For example, it was virtually impossible to write a PreCalculus lab on simplifying expressions involving radicals that actually gained anything from the use of the computer. The power of *Derive* actually added to this problem since the program does its simplification of such expressions over the complex numbers. That is, if a simplification holds for the real numbers, but not for complex numbers, then *Derive* must be "forced" to make the simplification. First-year college students are usually not ready for such distinctions.

A PreCalculus lab I am extremely pleased with is a lab that involved comparing the graphs of $f(x)$, $f(x)+C$, $f(x+C)$, $Cf(x)$, and $f(Cx)$. (Please see Lab 6 "Graphs of Functions" located at the end of this paper.) Not only did the lab go well at the time, but it made a tremendous impact later in the course when we studied the trigonometric functions. It has always been my experience that students have a hard time when they first encounter the topics of amplitude, period, and phase shift. I was amazed at how quickly they picked up these ideas when expressed in terms of the earlier lab. Even after six weeks they still remembered the lab and could apply it in a "new" situation!

In Calculus I was pleased with *Derive*'s ability to create and simplify sums. I took advantage of this during the presentation of the definition of integration in Calculus I, and in many of the applications of integration presented in Calculus II. (Please see Lab 17 "The Area Between Two Curves" located at the end of this paper.) Being able to (relatively) quickly calculate Reimann sums helped students feel comfortable with the concept. Furthermore, in many cases, *Derive* could evaluate the limit as n went to infinity of the Reimann sum. The fact that this answer agreed with the definite integral convinced students of the connection.

I experienced two difficulties in the labs. The first was dealing with various hardware problems. Experience and lots of calls to the Computer center have helped with this problem. Nevertheless, the first few weeks of the semester often provide extra "excitement," particularly if there have been any changes to the network between semesters. The second problem is maintaining the fine line between helping students move in the right direction and "giving away" the answers. This problem is particularly exacerbated at the end of a lab when the slower

workers are running out of time. If possible, I would suggest scheduling so as to alleviate the need to rush off immediately after the lab.

One unexpected problem with the implementation of these labs was the increased resentment of students **not** in the courses. The more classes scheduled in the computer labs, the less time there was available for other students to use the computer labs for their own assignments. As other courses, including courses offered by other departments, add computer labs and/or computer assignments, the demand for these computer labs continues to rise. Students who would never think to interrupt a class seem to have no problem interrupting a lab and demanding use of one of the computers.

Conclusions:
I am definitely glad that we added the computer labs to PreCalculus, Calculus I, and Calculus II. I hope we expand the use of Computer Algebra Systems to our other courses, at least on an informal basis. Students found having a second approach to the material aided their comprehension. Some students showed drastic improvement. However, the workload on the instructor/coordinator is greatly increased. This process should not be entered into without a firm commitment from both the Department and the College in general.

Sources:
[1] Larson, Hostetler, and Edwards, *Calculus Of a Single Variable*, 5th Ed., D. C. Heath and Company, 1994.

[2] Leithold, L., *Before Calculus: Functions, Graphs, and Analytic Geometry*, 2nd Ed., HarperCollins Publisher, 1989.

[3] Weida, R., *PreCalculus Computer Labs Using Derive*, Davis-Murphy Publishing, 1996.

[4] Weida, R., *Calculus Computer Labs Using Derive*, Davis-Murphy Publishing, 1996.

PreCalculus Lab 6 Graphs of Functions

Objective

In this lab you will compare the graphs of related functions and investigate the effect of adding or multiplying a constant value to the expression $f(x)$. Specifically, we want to study the relationship between the graph of $f(x)$ and the graphs of $f(x)+C$, $f(x+C)$, $Cf(x)$, and $f(Cx)$, where C is any real number.

<div style="border:3px solid black; text-align:center; padding:10px;">

New *Derive* Commands

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In this lab you will have more freedom then in previous labs. You should create your own examples to investigate the properties we are interested in. You should study enough different examples to be sure that you completely understand what is happening. It is important that you use several functions and that you try many values for the constant C. In particular, make sure that you use both positive and negative numbers and values whose magnitudes are both larger and smaller than one. When you have completely investigated one area, write a summery of the results in the most general form while being as precise as possible. We will do the first one together and you will do the others as part of the lab exercises.

The first relationship to study is the one between the graph of $f(x)+C$ and the graph of $f(x)$. Let's start with a simple function. Declare the function $f(x) = x^2$. Then enter the equation $y = f(x)$ and plot it. Now on the same graph we want to plot $y = f(x) + C$ for various values of C. Remember that each graph will be a different color and keep track of which graph is which. A few "nice" values to try first might be 1, 2, and ½. Switch back to the Algebra screen and enter the three equations, $y = f(x) + 1$, $y = f(x) + 2$, and $y = f(x) + ½$. Highlight the first equation, switch to the Graphics screen, and plot the equation. Switch back to the Algebra screen, highlight the next equation, switch back to the Graphics screen, and plot it. Do the same for the third equation. To see the four graphs redrawn in order, press the F9 and F10 function-keys simultaneously. The functions will be redrawn in the order in which you entered them. Your screen should look like Figure 1.
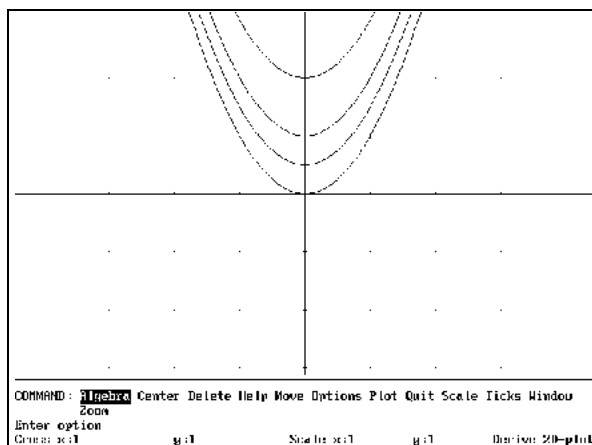
Figure 1

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Do you begin to see a pattern? Let's try some negative values. Adding -2 would be the same as subtracting 2. Instead of entering $y = f(x) + (-2)$ we'll use the equation $y = f(x) - 2$. Enter this equation in the Algebra screen and plot it in the Graphics screen. Do the same for the equations $y = f(x) - 1$ and $y = f(x) - ½$. OK, it looks like, when C is a positive number, the resulting graph is the original graph moved up C units. When C is a negative number, the graph is moved down C units. Whoops! What does that last sentence mean? For example, if C is -3, the graph is "moved down -3 units." How do you move negative units? OK, we better use absolute values so that the meaning is clear. When C is a negative number, the graph is moved down $|C|$ units.

We need to check this conclusion by trying other functions. Delete all the graphs in the Graphics screen. In the Algebra screen declare the function $f(x) = x^3$. Now, one at a time, plot each equation on the same graph. Remember you don't have to retype the equations since *Derive* will only use the latest version of the declared function. The "rule" we guessed above still seems to work. Maybe it only works for polynomials. Let's try a rational function. Delete all the graphs, declare the function $f(x) = \dfrac{1 + x}{x}$, and plot the equations. You'll probably want to zoom out a couple of times to get a better view. The rule still seems to hold. Next try the function $f(x) = \sqrt{x - 3}$. We've tried a few different functions and we used different values for C so we've probably covered all the cases. Now we need to write a conclusion for this case.

> The graph of the equation $y = f(x) + C$ is exactly the graph of the equation $y = f(x)$ moved vertically. If C is a positive number then the graph is moved C units up the y-axis. If C is a negative number then the graph is moved $|C|$ units down the y-axis.

As you complete this lab you will want to use a variety of functions. The following list of functions, while far from complete, provides some further examples you might consider.

$$f(x) = x^2 \qquad f(x) = x^3 \qquad f(x) = \frac{1}{x} \qquad f(x) = \frac{1}{x^2}$$

$$f(x) = \frac{x^2}{(x - 2)(x + 1)} \qquad\qquad f(x) = \sqrt{x}$$

$$f(x) = \sin x \qquad\qquad f(x) = \cos x \qquad\qquad f(x) = \frac{\sin x}{x}$$

$$f(x) = \frac{x + \sin(x^2)}{\sqrt{x}}$$

Exercises

1. Explore the relationship between the graphs of the equations $y = f(x+C)$ and $y = f(x)$. Be sure to use several functions and many possible values for the constant C including both positive and negative numbers and values whose magnitudes are both larger and smaller than one. When you have completely investigated this relationship, write a summery of the results in the most general form while being as precise as possible.

2. Explore the relationship between the graphs of the equations $y = Cf(x)$ and $y = f(x)$. Be sure to use several functions and many possible values for the constant C including both positive and negative numbers and values whose magnitudes are both larger and smaller than one. (I strongly suggest you specifically look at the value $C = -1$.) When you have completely investigated this relationship, write a summery of the results in the most general form while being as precise as possible.

3. Explore the relationship between the graphs of the equations $y = f(Cx)$ and $y = f(x)$. Be sure to use several functions and many possible values for the constant C including both positive and negative numbers and values whose magnitudes are both larger and smaller than one. (I strongly suggest you specifically look at the value $C = -1$.) When you have completely investigated this relationship, write a summery of the results in the most general form while being as precise as possible.

4. Describe the relationship between the graphs of the equations $y = 3f(2x-1) + 4$ and $y = f(x)$ as a combination of the above "movements." (I strongly suggest that you test your answer for specific functions using *Derive*.)

5. Let $g(x) = 2x^2 - 4x - 5$. Use the technique of completing the square to rewrite this function in the form $A(x+B)^2 + C$. Describe the relationship of the graph of the equations $y = g(x)$ and $y = x^2$. Sketch the graph of the equation $y = g(x)$ by hand.

Quick Review

In today's lab you studied the relationship between the graph of $f(x)$ and the graphs of $f(x)+C$, $f(x+C)$, $Cf(x)$, and $f(Cx)$, where C is any real number. You then saw that you could rewrite "messy" functions in terms of nicer functions. This will allow you to move and stretch the graphs of these nicer functions to get the graphs of the "messy" functions.

Calculus Lab 17 The Area Between Two Curves

Objective

In this lab you will review the formula for finding the area between two curves and use *Derive* to calculate these areas.

New *Derive* Commands

Recall that when we looked at the area **under** a curve, we first approximated the region by a collection of rectangles. The sum of the areas of these rectangles was approximately the area of the region. The actual area of the region is then the limit of this sum as the widths of all the rectangles go to zero. Luckily, this limit is nothing more than the definite integral and, usually we can use the Fundamental Theorem of Calculus to evaluate it.

Let's repeat this process for the area **between** two curves. Declare the two functions $f(x) = x$ and $g(x) = x^2 - 2$ and plot them on the same graph. It appears that the two graphs intersect at the points $x = -1$ and $x = 2$. Verify these intersections by solving the equation $f(x) = g(x)$. The area between these two curves is thus the area of the region between $x = -1$ and $x = 2$ that lies below the curve $f(x)$ and above the curve $g(x)$.

For our first approximation of the area of this region, let's partition the interval $[-1, 2]$ into six equal pieces. Each subinterval will have length $\Delta x = \dfrac{b - a}{n} = \dfrac{2 - (-1)}{6} = \dfrac{1}{2}$. Thus, the six subintervals are $[-1, -\frac{1}{2}]$, $[-\frac{1}{2}, 0]$, $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, $[1, 1\frac{1}{2}]$, and $[1\frac{1}{2}, 2]$. We must choose a point t_k in each subinterval. The "top" and "bottom" of the kth rectangle are then determined by $f(t_k)$ and $g(t_k)$, respectively. Figure 1 shows the resulting rectangles when t_k is chosen as the midpoint of the kth subinterval.

The midpoint of the first subinterval is $-\frac{3}{4}$. Therefore, the first rectangle has length $f(-\frac{3}{4}) - g(-\frac{3}{4})$ and width is $\Delta x = \frac{1}{2}$. Use *Derive* to compute the area of this rectangle. You should get $\dfrac{11}{32}$. The second rectangle has length $f(-\frac{1}{4}) - g(-\frac{1}{4})$ and width $\frac{1}{2}$. Compute its area.
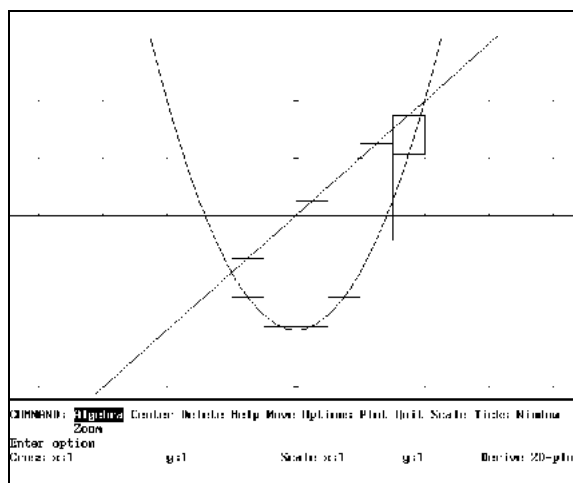
Figure 1

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You should get $\dfrac{27}{32}$. Continue this process for the other four rectangles and then add the six

areas together. You should get $\dfrac{73}{16}$ or 4.5625 as the sum of the areas of the rectangles.

       If we had chosen a different point for t_k then our approximation would have a different value. To see this, use the same partition as above but choose t_k to be the right endpoint of the interval. The first rectangle has length $f(-\frac{1}{2}) - g(-\frac{1}{2})$ and width ½. Therefore, its area is $\dfrac{5}{8}$.

Compute and add the areas of all six rectangles. This time the sum of the areas is $\dfrac{35}{8}$ or 4.375.

Which of these two approximations do you think is better? Why?

       To insure a better approximation of the area of the region we must decrease the width of the subintervals and therefore increase their number. Instead of continuing this process "by hand," let's automate as much of this process as possible. If we have n subintervals of equal

length, then each has width $\Delta x = \dfrac{b - a}{n} = \dfrac{2 - (-1)}{n} = \dfrac{3}{n}$. Therefore, the first subinterval is

$\left[-1, \; -1 + \dfrac{3}{n} \right]$; the second subinterval is $\left[-1 + \dfrac{3}{n}, \; -1 + 2\left(\dfrac{3}{n} \right) \right]$; the third subinterval is

$\left[-1 + 2\left(\dfrac{3}{n} \right), \; -1 + 3\left(\dfrac{3}{n} \right) \right]$; and the kth subinterval is $\left[-1 + (k-1)\left(\dfrac{3}{n} \right), \; -1 + k\left(\dfrac{3}{n} \right) \right]$.

Since it would be "messy" to denote the midpoint of this interval, we will choose t_k to be the right

endpoint of the interval. That is, we will choose $t_k = -1 + k\left(\dfrac{3}{n} \right)$.

      With this choice we see that the kth rectangle has length $f(t_k) - g(t_k)$ and width $\Delta x = \dfrac{3}{n}$.

Therefore, it has area $\left(f\left(-1 + \dfrac{3k}{n} \right) - g\left(-1 + \dfrac{3k}{n} \right) \right) \dfrac{3}{n}$. Adding the areas of the n

rectangles, we get $\displaystyle\sum_{k=1}^{n} \left(f\left(-1 + \dfrac{3k}{n} \right) - g\left(-1 + \dfrac{3k}{n} \right) \right) \dfrac{3}{n}$ as the approximation of the area

of the region. To create this sum, first enter the expression $(f(-1+3k/n)-g(-1+3k/n))(3/n)$.
Then choose Calculus, Sum and press Enter three times. The first time is for the correct
expression, the second time is for the correct variable, and the third time is for the correct limits.
Finally, use the F3 function-key and declare this summation as the function $h(n)$. The last three
expressions on your screen should look like Figure 2. The function $h(n)$ represents the sum of
the areas of the n rectangles with equal width and using the right endpoint to determine the
height.

To check our computations evaluate $h(6)$. You should get the same answer as when we did this "by hand," namely $\dfrac{35}{8}$ or 4.375. Now evaluate $h(25)$, the sum of the areas using 25 rectangles. Similarly, evaluate $h(100)$, and $h(1000)$. As you can see, as we increase the number of rectangles, the area approaches a value of 4.5. (If you increase the number of digits of accuracy, you will see that it never quite reaches this value.)

$$\left[F\left[-1 + \frac{3\,k}{n}\right] - G\left[-1 + \frac{3\,k}{n}\right] \right] \frac{3}{n}$$

$$\sum_{k=1}^{n} \left[F\left[-1 + \frac{3\,k}{n}\right] - G\left[-1 + \frac{3\,k}{n}\right] \right] \frac{3}{n}$$

$$H(n) := \sum_{k=1}^{n} \left[F\left[-1 + \frac{3\,k}{n}\right] - G\left[-1 + \frac{3\,k}{n}\right] \right] \frac{3}{n}$$

Figure 2

To get the exact value of the area of the region between the two curves you must evaluate $\displaystyle\lim_{n \to \infty} h(n)$. Use *Derive* to do so. (Type in the function name and choose Calculus, Limit ...) The area between the curves is 4.5. Remember that $\displaystyle\lim_{n \to \infty} h(n)$ really represents

$$\lim_{n \to \infty} \sum_{k=1}^{n} \big(f(t_k) - g(t_k)\big)\Delta x.$$ This is just the limit of a Riemann Sum and is therefore equal to

$$\int_{-1}^{2} \big(f(x) - g(x)\big)\,dx.$$ Use *Derive* to evaluate this integral. (Type in the expression and choose Calculus, Integrate ...) Of course, you will get the same answer.

You're probably wondering why we went to all the trouble to evaluate various sums of areas when the integral was so much easier to compute. The first reason to do so is to reinforce the definition of integration. Too many students confuse the integral and the shortcut provided by the Fundamental Theorem. Secondly, this approach will help you understand (and remember) the various formulas in this chapter. Thirdly, this process should give you some insight into numerical integration.

For our next example, consider the two functions $f(x) = x^3 + x^2 - x - 1$ and $g(x) = x^2 - 1$. Declare and graph these functions. Then verify that the graphs intersect at the points $x = -1$, $x = 0$, and $x = 1$. It might be tempting to think that the area between these two curves is equal to $$\int_{-1}^{1} \big(f(x) - g(x)\big)\,dx,$$ but it's not. (Evaluate this integral. What do you get? Why?) Remember, in the formula for area between two curves, the function f was the function that formed the "top" of the rectangle and the function g formed the "bottom" of the rectangle. Which of the two given functions in this example is the "top" one? From the graph we see that f is the "top" function from $x = -1$ to $x = 0$, but g is the "top" function from $x = 0$ to $x = 1$. We'll

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have to break this region into two pieces and use two integrals. The area of the region is given by

$$\int_{-1}^{0} \big(f(x) - g(x)\big)\, dx \;+\; \int_{0}^{1} \big(g(x) - f(x)\big)\, dx.$$ Evaluate these integrals to verify that the area

of the region is ½.

Exercises

1. Consider the region between the curves $y = x^2$ and $y = x^3$.
a) Approximate the area of this region by partitioning the appropriate interval into 4 equal parts.
b) Approximate the area of this region by partitioning the appropriate interval into 20 equal parts.
c) Approximate the area of this region by partitioning the appropriate interval into 100 equal parts.
d) Find the exact area of the region by taking the limit of an appropriate Riemann Sum.
e) Use integration to find the exact area of the region.

2. Find the area of the region between the two curves given by $f(x) = x^3 - 2x^2$ and $g(x) = 2x^2 - 3x$.

3. Find the area of the region between the two curves given by $f(x) = x^4 - 3x^3 - x^2 + 3x + 1$ and $g(x) = 2 - x^2$.

4. Write a paragraph (at least five sentences) describing how to find the area between two curves.

5. Find the area of the region bounded by the curves $y = x + 2$, $y = 6 - 3x$, and $3y = 2 - x$.

6. Find a horizontal line, $y = k$, which divides the area between $y = x^2$ and $y = 9$ into two equal parts.

Quick Review

In this lab you worked with the definition of integration to find the area of a region bounded by two or more curves. Then you used the Fundamental Theorem of Calculus to evaluate these integrals. Hopefully, this reinforced both the definition of integration and the "formula" for finding the area between two curves.