# Graphics Calculator Applications on 4 - D Constructs

The measure of an $n$ - D geometrical object, $m(g_n)$, is the real number, $r$, of $n$ - D unit cubes, $u^n$, that tessellate (tiles with no laps or gaps) the geometric object. $m(g_n) = r \ \ u^n$. Some $n$ - D geometric figures, $p_u$, can be countably tessellated by $n$ - D constructs, $c_n$, using the vertices. $p_n = \bigcup_{i=1}^{k} m(c_{ni})$. The measure of such objects, $m(p_n)$, is the sum of the measures of the tessellating constructs. $m(p_n) = \sum_{i=1}^{k} m(c_{ni})$.

The measure of any $n$ - D construct, $c_n$, in terms of the distances between the vertices of the construct is expressed as: $m(c_n) = \frac{1}{n!}\sqrt{\frac{1}{2^n} D_{nxn}} \quad u^n$. $D_{nxn}$ is a determinate such that each numerical entry in the $i$ th row and $j$ th column, $a_{ij}$, is determined as:
$a_{ij} = (num(d(V_{o,}V_i)))^2 + (num(d(V_{o,}V_j)))^2 - (num(d(V_{i,}V_j)))^2$. $num(d(V_{i,}V_j))$ is the numerical part of the measure of the distance between vertex $i$ and vertex $j$. $V_0$ is merely a reference vertex of the construct to all other vertices of the construct. **Diagram 1** depicts a typical trigonal paradigm for determining $a_{ij}$ .

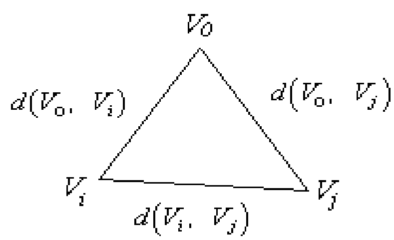


**Diagram 1**. The trigonal paradigm for determining $a_{ij}$ .

The expression $m(c_n) = \frac{1}{n!}\sqrt{\frac{1}{2^n} D_{nxn}} \quad u^n$ was proven to represent the measure of any $n$ - D construct in terms of the measures of the distances between the vertices by the author and Eugene Curtin and presented to the Mathematics Department Colloquium, Southwest Texas State University on May 24, 1995. The author had proven the formula for 0 - D , 1 - D , 2 - D , and 3 - D in early Spring 1991 in preparation for a statewide convention for faculty of gifted/talented students and presented later that same year. These “real” space proofs will follow as a matter of demonstration.

The authors of the paper have examined the measurable attributes of tetrahedrons and have presented calculator applications on such at previous meetings of this society. A unique tetrahedron, the smallest tetrahedron having integral sides and integral volume, has been the basis of these presentations. This same tetrahedron will serve as a pedestal

for generating a 4 - D construct with consecutive integers. That is, we will pick a point in
4 - D space, not in the 3 - D space of the pedestal tetrahedron to serve as an initial reference vertex, $V_0$ . We will simply select the next four integral lengths to locate $V_0$ from the vertices of the pedestal tetrahedron. Since there are several such choices for a 4 -D construct, $c_4$ , we will select as an example for this paper the sequence that we happened to try first, experimentally. This choice is depicted below in **Diagram 2**. Keep in mind that the diagram is an image of a 4 - D construct on a 2 - D space, so appearances may be somewhat obscure and no attempt was made to draw the diagram to scale.

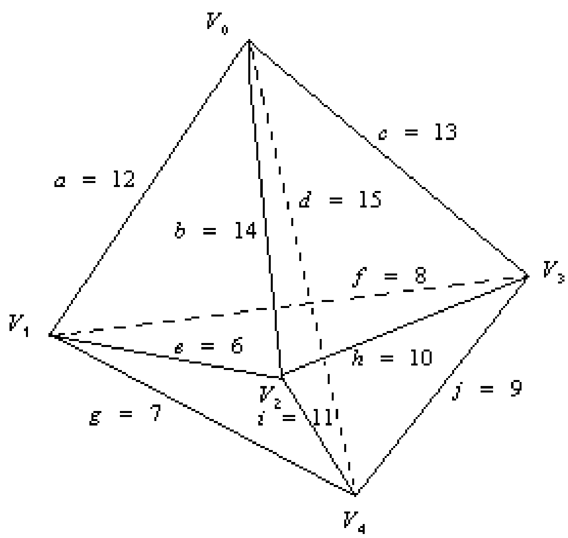


**Diagram 2.** A 2 - D image of a 4 - D construct based upon a unique tetrahedron and having consecutive integral edges.

The 4 - D construct, called a *pentatope* and symbolized as $c_4$, or specifically, $c_4$(12,14, 13, 15, 6, 8, 7, 10, 11, 9), has edges as follows:

$$d(V_0,V_1) = a\ u = 12\ u$$
$$d(V_0,V_2) = b\ u = 14\ u$$
$$d(V_0,V_3) = c\ u = 13\ u$$
$$d(V_0,V_4) = d\ u = 15\ u$$
$$d(V_1,V_2) = e\ u = 6\ u$$
$$d(V_1,V_3) = f\ u = 8\ u$$
$$d(V_1,V_4) = g\ u = 7\ u$$
$$d(V_2,V_3) = h\ u = 10\ u$$
$$d(V_2,V_4) = i\ u = 11\ u$$
$$d(V_3,V_4) = j\ u = 9\ u$$

The measure (sometimes generically referred to as volume) of any 4 - D construct is

$$m(c_4) = \frac{1}{4!}\sqrt{\frac{1}{2^4} D_{4x4}} \quad u^4 \ .$$

The formula simply counts the 4 - D unit cubes, called unit tesseracts, which tessellate the region of 4 - D space that is bounded by the five 3 - D constructs, facets: $c_{31}(V_0,V_1,V_2,V_3)$, $c_{32}(V_0,V_1,V_2,V_4)$, $c_{33}(V_0,V_1,V_3,V_4)$, $c_{34}(V_0,V_2,V_3,V_4)$, $c_{35}(V_1,V_2,V_3,V_4)$, which is why a $c_4$ is referred to as a *pentatope*.

$$D_{4x4} \quad = \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$\begin{aligned}
a_{11} &= (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_1)))^2 - (num(d(V_{1,}V_1)))^2 \\
&= a^2 + a^2 - 0^2 \\
&= 2a^2 \\
a_{12} &= (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_2)))^2 - (num(d(V_{1,}V_2)))^2 \\
&= a^2 + b^2 - e^2 \\
&= a_{21} \\
a_{13} &= (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_3)))^2 - (num(d(V_{1,}V_3)))^2 \\
&= a^2 + c^2 - f^2 \\
&= a_{31} \\
a_{14} &= (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_4)))^2 - (num(d(V_{1,}V_4)))^2 \\
&= a^2 + d^2 - g^2 \\
&= a_{41} \\
a_{22} &= (num(d(V_{0,}V_2)))^2 + (num(d(V_{0,}V_2)))^2 - (num(d(V_{2,}V_2)))^2 \\
&= b^2 + b^2 - 0^2 \\
&= 2b^2 \\
a_{23} &= (num(d(V_{0,}V_2)))^2 + (num(d(V_{0,}V_3)))^2 - (num(d(V_{2,}V_3)))^2 \\
&= b^2 + c^2 - h^2 \\
&= a_{32}
\end{aligned}$$

$$a_{24} = (num(d(V_{0,}V_2)))^2 + (num(d(V_{0,}V_4)))^2 - (num(d(V_{2,}V_4)))^2$$
$$= \quad b^2 \quad + \quad d^2 \quad - \quad i^2$$
$$= \quad a_{42}$$

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$$a_{33} = (num(d(V_{0,}V_3)))^2 + (num(d(V_{0,}V_3)))^2 - (num(d(V_{3,}V_3)))^2$$
$$= \quad c^2 \quad + \quad c^2 \quad - \quad 0^2$$
$$= \quad 2c^2$$
$$a_{34} = (num(d(V_{0,}V_3)))^2 + (num(d(V_{0,}V_4)))^2 - (num(d(V_{3,}V_4)))^2$$
$$= \quad c^2 \quad + \quad d^2 \quad - \quad j^2$$
$$= \quad a_{43}$$
$$a_{44} = (num(d(V_{0,}V_4)))^2 + (num(d(V_{0,}V_4)))^2 - (num(d(V_{4,}V_4)))^2$$
$$= \quad d^2 \quad + \quad d^2 \quad - \quad 0^2$$
$$= \quad 2d^2$$

$$m(c_4) = m(a,b,c,d,e,f,g,h,i,j)$$
$$= \frac{1}{4!}\sqrt{\frac{1}{2^4} D_{4x4}} \quad u^4$$
$$= \frac{1}{24}\sqrt{\frac{1}{16}\begin{vmatrix} 2a^2 & a^2+b^2-e^2 & a^2+c^2-f^2 & a^2+d^2-g^2 \\ a^2+b^2-e^2 & 2b^2 & b^2+c^2-h^2 & b^2+d^2-i^2 \\ a^2+c^2-f^2 & b^2+c^2-h^2 & 2c^2 & c^2+d^2-j^2 \\ a^2+d^2-g^2 & b^2+d^2-i^2 & c^2+d^2-j^2 & 2d^2 \end{vmatrix}}$$

which is programmable.

$m(12,14,13,15,6,8,7,10,11,9) \approx 130.9556223 \ u^4$ to the nearest $10^{-7}$.

The pentatope has five facets and each facet is a 3 - D construct, $c_3$. The measure of any $c_3$ in terms of the distances between the vertices or edges is

$$m(c_3) = \frac{1}{3!}\sqrt{\frac{1}{2^3} D_{3x3}} \quad u^3 .$$

$$D_{3x3} \quad = \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

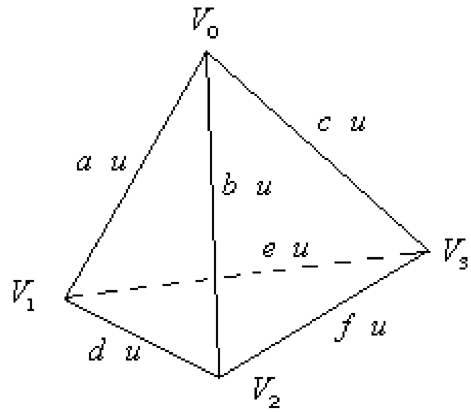



**Diagram 3.** A 2 - D image of a 3 - D construct.

Diagram 3 may help in representing the numerical entries for $D_{3x3}$ .

$$a_{11} = (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_1)))^2 - (num(d(V_{1,}V_1)))^2$$
$$= a^2 + a^2 - 0^2$$
$$= 2a^2$$
$$a_{12} = (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_2)))^2 - (num(d(V_{1,}V_2)))^2$$
$$= a^2 + b^2 - d^2$$
$$= a_{21}$$
$$a_{13} = (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_3)))^2 - (num(d(V_{1,}V_3)))^2$$
$$= a^2 + c^2 - e^2$$
$$= a_{31}$$
$$a_{22} = (num(d(V_{0,}V_2)))^2 + (num(d(V_{0,}V_2)))^2 - (num(d(V_{2,}V_2)))^2$$
$$= b^2 + b^2 - 0^2$$
$$= 2b^2$$
$$a_{23} = (num(d(V_{0,}V_2)))^2 + (num(d(V_{0,}V_3)))^2 - (num(d(V_{2,}V_3)))^2$$
$$= b^2 + c^2 - f^2$$
$$= a_{32}$$
$$a_{33} = (num(d(V_{0,}V_3)))^2 + (num(d(V_{0,}V_3)))^2 - (num(d(V_{3,}V_3)))^2$$
$$= c^2 + c^2 - 0^2$$
$$= 2c^2$$

$$m(c_3) = m(a,b,c,d,e,f)$$
$$= \left(\frac{1}{3!}\right)\sqrt{\frac{1}{2^3}D_{3x3}}\quad u^3$$

$$= \left(\frac{1}{6}\right)\sqrt{\frac{1}{8}\begin{vmatrix} 2a^2 & a^2+b^2-d^2 & a^2+c^2-e^2 \\ a^2+b^2-d^2 & 2b^2 & b^2+c^2-f^2 \\ a^2+c^2-e^2 & b^2+c^2-f^2 & 2c^2 \end{vmatrix}}\ u^3$$

which is programmable.

Dissecting $c_4$ we have the following five tessellating tetrahedrons, which are facets of $c_4$: $c_{31}$ (12, 14, 13, 6, 8, 10) is depicted in **Diagram 4**.

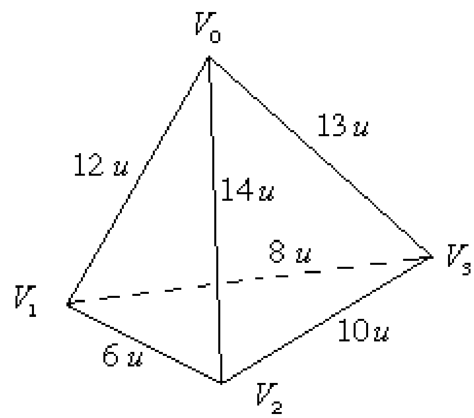



**Diagram 4**. $C_{31}$ (12, 14, 13, 6, 8, 10). One of the tetrahedrons which tessellate the chosen $C_4$ .

$m(c_{31}) = m$ (12, 14, 13, 6, 8, 10) = 93.3914997 $u^3$ to the nearest $10^{-7}$ .

$c_{32}$ (12, 14, 15, 6, 7, 11) is depicted in **Diagram 5**.

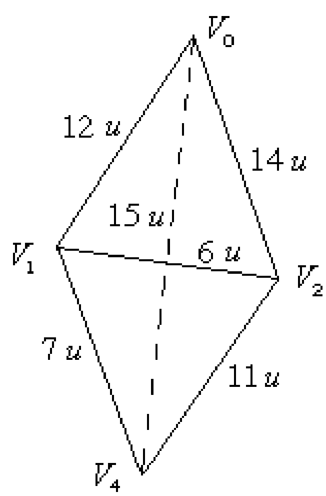



**Diagram 5**. $C_{32}$ (12, 14, 15, 6, 7, 11), one of the tetrahedrons which tessellate the chosen $C_4$ .

$m(c_{32}) = m$ (12, 14, 15, 6, 7, 11) = 72.7247474 $u^3$ to the nearest $10^{-7}$ .

$c_{33}$ (12, 13, 15, 8, 7, 9) is depicted in **Diagram 6**.

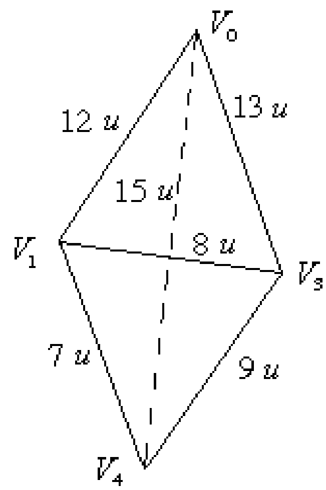



**Diagram 6**. $c_{33}$ (12, 13, 15, 8, 7, 9), one of the tetrahedrons which tessellate the chosen $c_4$ .

$m(c_{33}) = m(12, 13, 15, 8, 7, 9) = 101.3409742 \ u^3$ to the nearest $10^{-7}$ .

$c_{34}$ (14, 13, 15, 10, 11, 9) is depicted in **Diagram 7**.

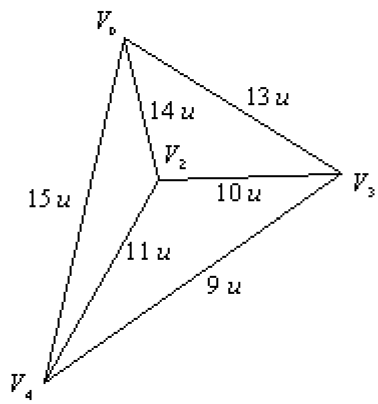



**Diagram 7**. $c_{34}$ (14, 13, 15, 10, 11, 9), one of the tetrahedrons which tessellate the chosen $c_4$ .

$m(c_{34}) = m(14, 13, 15, 10, 11, 9) = 176.4336317 \ u^3$ to the nearest $10^{-7}$ .

$c_{35}$ (6, 8, 7, 10, 11, 9) is depicted in **Diagram 8**.

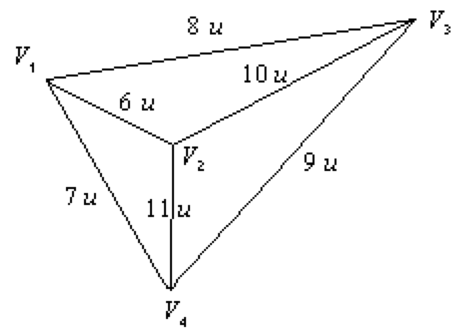



**Diagram 8**. $c_{35}$ (6, 8, 7, 10, 11, 9), one of the tetrahedrons which tessellate the chosen $c_4$ .

$$m(c_{35}) = m\,(6,\,8,\,7,\,10,\,11,\,9) \;=\; 48\;\; u^3 .$$

$$\sum_{i=1}^{5} m(c_{3i}) = \;\; 491.896853 \;\; u^3 \text{ to the nearest } 10^{-6} \ .$$

A $c_4$ is tessellated with ten 2 - D constructs, trigons, $c_2$ . The measure of any $c_2$ in terms of the distances between the vertices or edges is:

$$m(c_2) = \frac{1}{2!}\sqrt{\frac{1}{2^2}D_{2x2}} \quad u^2$$

$$D_{2x2} \quad = \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

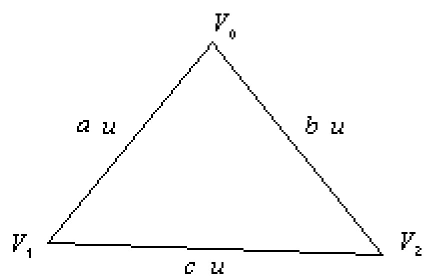



**Diagram 9**. A 2 - D construct, trigon, $c_2$ .

$$a_{11} = \big(num\big(d\,(V_{0,}V_1\,)\big)\big)^2 + \big(num\big(d\,(V_{0,}V_1\,)\big)\big)^2 - \big(num\big(d\,(V_{1,}V_1\,)\big)\big)^2$$
$$= \quad a^2 \;\; + \;\; a^2 \;\; - \;\; 0^2$$
$$= \quad 2a^2$$
$$a_{12} = \big(num\big(d\,(V_{0,}V_1\,)\big)\big)^2 + \big(num\big(d\,(V_{0,}V_2\,)\big)\big)^2 - \big(num\big(d\,(V_{1,}V_2\,)\big)\big)^2$$
$$= \quad a^2 \;\; + \;\; b^2 \;\; - \;\; c^2$$
$$= \quad a_{21}$$
$$a_{22} = \big(num\big(d\,(V_{0,}V_2\,)\big)\big)^2 + \big(num\big(d\,(V_{0,}V_2\,)\big)\big)^2 - \big(num\big(d\,(V_{2,}V_2\,)\big)\big)^2$$

$$= \quad b^2 \quad + \quad b^2 \quad - \quad 0^2$$
$$= \quad 2b^2$$

$$m(c_2) = m\,(\text{a, b, c})$$
$$= \quad \frac{1}{2\;!}\sqrt{\frac{1}{2^2}\;D_{2x2}}\quad u^2$$
$$= \quad \frac{1}{2}\sqrt{\frac{1}{4}\begin{vmatrix} 2a^2 & a^2 + b^2 - c^2 \\ a^2 + b^2 - c^2 & 2b^2 \end{vmatrix}}\; u^2$$

which is programmable.

Each $c_{3i}$ facet of $p_4$ has four facets which are subfacets of $c_4$ . The subfacets of $c_4$ are the tessellating $c_{2k}$ facets of each $c_{3i}$ .

The facets of $c_{31}$ are determined by the following four triplets of vertices and each triplet of vertices is paired with the corresponding triplets of edges from the specific $c_4$ :

$(V_0,V_1,V_2)$ - (12, 14, 6) $\qquad$ $(V_0,V_1,V_3)$ - (12, 13, 8)
$(V_0,V_2,V_3)$ - (14, 13, 10) $\qquad$ $(V_1,V_2,V_3)$ - (6, 8, 10) .

The facets of $c_{32}$ are determined by the following four triplets of vertices and each triplet of vertices is paired with the corresponding triplets of edges from the specific $c_4$ :

$(V_0,V_1,V_2)$ - (12, 14, 6) $\qquad$ $(V_0,V_1,V_4)$ - (12, 15, 7)
$(V_0,V_2,V_4)$ - (14, 15, 11) $\qquad$ $(V_1,V_2,V_4)$ - (6, 7, 11) .

Notice that $(V_0,V_1,V_2)$ is common to facets of $c_{31}$ and $c_{32}$ and should not be used twice since any such usage would violate the essence of tessellation.

The facets of $c_{33}$ are determined by the following triplets of vertices and each triplet of vertices is paired with the corresponding triplets of edges from the specific $c_4$ :

$(V_0,V_1,V_3)$ - (12, 13, 8) $\qquad$ $(V_0,V_1,V_4)$ - (12, 15, 7)
$(V_0,V_3,V_4)$ - (13, 15, 9) $\qquad$ $(V_1,V_3,V_4)$ - (8, 7, 9) .

Notice that $(V_0,V_1,V_3)$ is common to facets of $c_{31}$ and $c_{33}$ and $(V_0,V_1,V_4)$ is common to facets of $c_{31}$ and $c_{33}$ and neither should be used twice in determining the tessellating $c_{2k}$ subfacets of $c_4$ .

The facets of $c_{34}$ are determined by the following triplets of vertices and each triplet of vertices is paired with the corresponding triplets of edges from the specific $c_4$ :

$(V_0, V_2, V_3)$ - (14, 13, 10) $(V_0, V_2, V_4)$ - (14, 15, 11)

$(V_0, V_3, V_4)$ - (13, 15, 9) $(V_2, V_3, V_4)$ - (10, 11, 9) .

Notice that $(V_0, V_2, V_3)$ is common to facets of $c_{31}$ and $c_{34}$ and $(V_0, V_2, V_4)$ is common to facets of $c_{23}$ and $c_{34}$ and $(V_0, V_3, V_4)$ is common to facets of $c_{33}$ and $c_{34}$ and none of these facets should be used twice in determining the tessellating $c_{2k}$ subfacets of $c_4$ .

The facets of $c_{35}$ are determined by the following triplets of vertices and each triplet of vertices is paired with the corresponding triplets of edges from the specific $c_4$ :

$(V_1, V_2, V_3)$ - (6, 8, 10) $(V_1, V_2, V_4)$ - (6, 7, 11)

$(V_1, V_3, V_4)$ - (8, 7, 9) $(V_2, V_3, V_4)$ - (10, 11, 9) .

Notice that $(V_1, V_2, V_3)$ is common to facets of $c_{31}$ and $c_{35}$ ,

$(V_1, V_2, V_4)$ is common to $c_{32}$ and $c_{35}$

$(V_1, V_3, V_4)$ is common to $c_{33}$ and $c_{35}$

and $(V_2, V_3, V_4)$ is common to $c_{34}$ and $c_{35}$ .

So, none of these facets of $c_{35}$ contributes to any new subfacets of $c_4$ .

The measures of the subfacets tessellating $c_{2k}$ of $c_4$ are to the nearest $10^{-7}$:

$m(c_{21}(12,14,6)) =$ 35.7770876 $u^2$

$m(c_{22}(12,13,8)) =$ 46.9993351 $u^2$

$m(c_{23}(14,13,10)) =$ 62.3853949 $u^2$

$m(c_{24}(6,8,10)) =$ 24 $u^2$

$m(c_{25}(12,15,7)) =$ 41.2310563 $u^2$

$m(c_{26}(14,15,11)) =$ 73.4846923 $u^2$

$m(c_{27}(6,7,11)) =$ 18.9736660 $u^2$

$m(c_{28}(13,15,9)) =$ 58.1651743 $u^2$

$m(c_{29}(8,7,9)) =$ 26.8328157 $u^2$

$m(c_{210}(10,11,9)) =$ 42.4264069 $u^2$

$$\sum_{k=1}^{10} m(c_{2k}) = \quad 430.2756291 \ u^2$$ , to the nearest $10^{-7}$.

A $c_4$ has ten edges and each edge is a line segment or a 1 - D construct, $c_1$. The measure of any $c_1$ in terms of the distances between the vertices or the length of an edge is:

$$m(c_1) = \frac{1}{1!}\sqrt{\frac{1}{2^1}D_{1x1}} \quad u^1 \ .$$

$$D_{1x1} \quad = \quad |a_{11}| \ .$$

**Diagram 10** depicts a 1 - D construct, $c_1$ .


**Diagram 10**. A 1 - D construct, line segment, edge, $c_1$ .

$$\begin{aligned} a_{11} &= (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_1)))^2 - (num(d(V_{1,}V_1)))^2 \\ &= a^2 + a^2 - 0^2 \\ &= 2a^2 \end{aligned}$$

$$\begin{aligned} m(c_1) &= m(a) \\ &= \frac{1}{1\ !}\sqrt{\frac{1}{2^1}\ D_{1x1}} \quad u^1 \\ &= \frac{1}{1}\sqrt{\frac{1}{2}\ |2a^2|} \quad u^1 \\ &= a \quad u^1 \ . \end{aligned}$$

Each of the ten edges might be referred to as a sub-subfacet of $c_4$ or sub $^2$ -facet of $c_4$ .

$$\begin{aligned} m(c_{11}) &= \quad 12 \ u^1 \\ m(c_{12}) &= \quad 14 \ u^1 \\ m(c_{13}) &= \quad 13 \ u^1 \\ m(c_{14}) &= \quad 15 \ u^1 \\ m(c_{15}) &= \quad 6 \ u^1 \\ m(c_{16}) &= \quad 8 \ u^1 \\ m(c_{17}) &= \quad 7 \ u^1 \\ m(c_{18}) &= \quad 10 \ u^1 \\ m(c_{19}) &= \quad 11 \ u^1 \\ m(c_{110}) &= \quad 9 \ u^1 \end{aligned}$$

$$\sum_{j=1}^{10} m(c_{1j}) = \quad 105 \ u^1$$

A $c_4$ has five vertices and each vertex is a point or a 0 - D construct, $c_0$ . The measure of any $c_0$ is:

$$m(c_0) = \frac{1}{0!}\sqrt{\frac{1}{2^0} D_{0x0}} \quad u^0$$
$$= \quad 1 \quad u^0 \ .$$

Each of the five vertices might be referred to as a sub-sub-subfacet of $c_4$ , or sub $^3$ -facet of $c_4$ .

$$m(c_{01}) = 1 \quad u^0$$
$$m(c_{02}) = 1 \quad u^0$$
$$m(c_{03}) = 1 \quad u^0$$
$$m(c_{04}) = 1 \quad u^0$$
$$m(c_{05}) = 1 \quad u^0$$

$$\sum_{g=1}^{5} m(c_{0g}) = \quad 5 \quad u^0 \ .$$

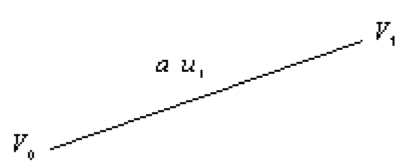
**The following constitute "real" space proofs** for counting the number of cubes that tessellate "real" constructs:

**The size of a point**, a 0 - D construct, $c_0$ .

$$m(c_0) = 1 \qquad u^0$$ (a point tessellates a point and a point is also a 0 - D cube, $u^0$ )

$$= \quad \frac{1}{0\ !}\sqrt{\frac{1}{2^0} \ D_{0x0}} \quad u^0 \ .$$

**The size of a line segment**, a 1 - D construct, $c_1$ .
Suppose the measure of a line segment determined by vertices $V_0$ and $V_1$ is $a \ u^1$ .



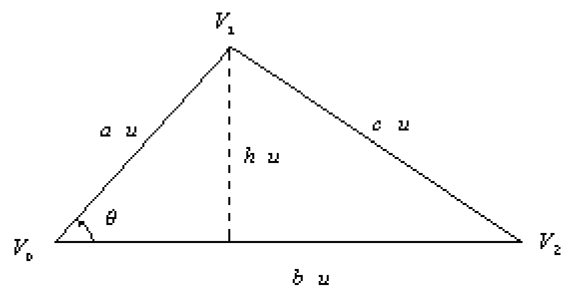

$$m(c_1) = a \quad u^1$$
$$= \quad \frac{1}{1\ !}\sqrt{\frac{1}{2^1} \ \left|2a^2\right|} \quad u^1$$

$$= \frac{1}{1\ !}\sqrt{\frac{1}{2^1}\ D_{1x1}}\quad u^1$$

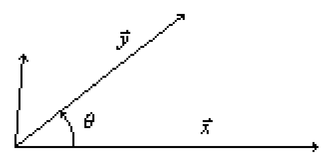
where $D_{1x1} = |a_{11}|$ and

$$a_{11} = (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_1)))^2 - (num(d(V_{1,}V_1)))^2$$
$$= \quad 2a^2$$

**The area of a trigon**, a 2 - D construct, $c_2$ :

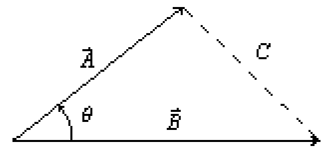



$$m(c_2) = \frac{bh}{2} = \frac{b(a(\sin(\theta)))}{2} \qquad \text{where } \sin(\theta) = \frac{h}{a} \text{ or } h = a(\sin(\theta)),$$
$$= \frac{(ba)(\sin(\theta))}{2}\ .$$




The cross product of vectors, $\vec{X}$, $\vec{Y}$, is $\vec{X} \times \vec{Y}$, where $\vec{X} \times \vec{Y}$ is a vector at a right angle to the plane determined by $\vec{X}$ and $\vec{Y}$. $\vec{Y} \times \vec{X}$ is a vector of the same magnitude but in the opposite direction.

The magnitude of $\vec{X} \times \vec{Y}$ is $\left\|\vec{X} \times \vec{Y}\right\| = \left\|\vec{X}\right\|\left\|\vec{Y}\right\|(\sin(\theta))$

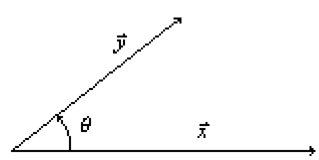



$$\left\|\vec{A}\right\| = a,\ \left\|\vec{B}\right\| = b,\ \left\|\vec{B} \times \vec{A}\right\| = \left\|\vec{A} \times \vec{B}\right\|$$
$$= \left\|\vec{B}\right\|\left\|\vec{A}\right\|(\sin(\theta))$$
$$= ba(\sin(\theta))\ .$$

From above,

$$m(c_2) = \frac{\left\|\vec{A} \times \vec{B}\right\|}{2}\ u^2 \qquad \text{and clearly } \left\|\vec{A} \times \vec{B}\right\| \geq 0\,,$$
$$= \frac{1}{2}\sqrt{\left\|\vec{A} \times \vec{B}\right\|^2}\ u^2$$

$$\left(\left\|\vec{A} \times \vec{B}\right\|^2\right) = \left\|\vec{A}\right\|^2\left\|\vec{B}\right\|^2\left(\sin^2(\theta)\right)$$
$$= \left\|\vec{A}\right\|^2\left\|\vec{B}\right\|^2\left(1 - \cos^2(\theta)\right)$$
$$= \left\|\vec{A}\right\|^2\left\|\vec{B}\right\|^2 - \left\|\vec{A}\right\|^2\left\|\vec{B}\right\|^2\left(\cos^2(\theta)\right)$$
$$= \left\|\vec{A}\right\|^2\left\|\vec{B}\right\|^2 - \left(\left\|\vec{A}\right\|\left\|\vec{B}\right\|(\cos(\theta))\right)^2$$



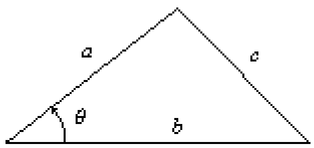

The Dot Product of vectors, $\vec{X}$ and $\vec{Y}$, is $\vec{X} \bullet \vec{Y} = \left\|\vec{X}\right\|\left\|\vec{Y}\right\|(\cos(\theta))$ .

From above,

$$\left(\left\|\vec{A} \times \vec{B}\right\|^2\right) = \left\|\vec{A}\right\|^2\left\|\vec{B}\right\|^2 - \left(\vec{A} \bullet \vec{B}\right)^2 \ , \quad \text{LaGrange Identity.}$$

Also from above,

$$m(c_2) = \frac{1}{2}\sqrt{\|\vec{A}\|^2\|\vec{B}\|^2 - \left(\vec{A}\bullet\vec{B}\right)^2}\ \ u^2 \ .$$




Law of Cosines, $\cos(\theta) = \dfrac{a^2 + b^2 - c^2}{2ab}$ .

$$2ab(\cos(\theta)) = a^2 + b^2 - c^2 \ .$$

$$ab(\cos(\theta)) = \frac{a^2 + b^2 - c^2}{2} \ .$$

So,

$$\|\vec{A}\|\|\vec{B}\|(\cos(\theta)) = \frac{a^2 + b^2 - c^2}{2}$$

$$\vec{A} \bullet \vec{B} = \frac{a^2 + b^2 - c^2}{2}$$

and

$$\|\vec{A}\| = a \ , \quad \|\vec{B}\| = b \ .$$

And, from before,

$$m(c_2) = \frac{1}{2}\sqrt{a^2b^2 - \left(\frac{a^2 + b^2 - c^2}{2}\right)^2}\ \ u^2$$

$$= \frac{1}{2}\sqrt{\frac{4a^2b^2 - \left(a^2 + b^2 - c^2\right)^2}{4}}\ \ u^2$$

$$= \frac{1}{2}\sqrt{\frac{1}{4}\left(2a^2 2b^2 - \left(a^2 + b^2 - c^2\right)^2\right)}\ \ u^2$$

$$= \frac{1}{2\ !}\sqrt{\frac{1}{2^2}\begin{vmatrix} 2a^2 & a^2 + b^2 - c^2 \\ a^2 + b^2 - c^2 & 2b^2 \end{vmatrix}}\ \ u^2$$

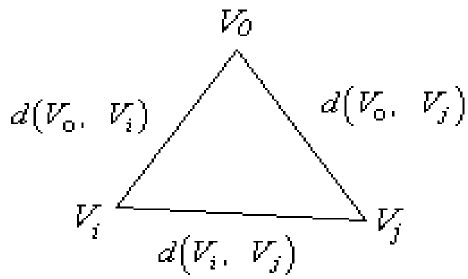
$$= \frac{1}{2\ !}\sqrt{\frac{1}{2^2}\ D_{2x2}}\ \ u^2$$

where $a_{11} = 2a^2$

$$= a^2 + a^2 - 0^2$$

$$= \left(num\left(d\left(V_{0,}V_1\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_1\right)\right)\right)^2 - \left(num\left(d\left(V_{1,}V_1\right)\right)\right)^2$$

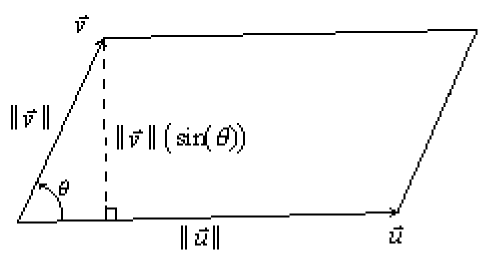
$$\begin{aligned} a_{12} &= a_{21} \\ &= a^2 + b^2 - c^2 \\ &= \left(num\left(d\left(V_{0,}V_1\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_2\right)\right)\right)^2 - \left(num\left(d\left(V_{1,}V_2\right)\right)\right)^2 \\ a_{22} &= 2b^2 \\ &= b^2 + b^2 - 0^2 \\ &= \left(num\left(d\left(V_{0,}V_2\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_2\right)\right)\right)^2 - \left(num\left(d\left(V_{2,}V_2\right)\right)\right)^2 \end{aligned}$$




$$a_{ij} = \left(num\left(d\left(V_{o,}V_i\right)\right)\right)^2 + \left(num\left(d\left(V_{o,}V_j\right)\right)\right)^2 - \left(num\left(d\left(V_{i,}V_j\right)\right)\right)^2$$

**Volume of a parallelepiped** and of a 3 - D construct, $c_3$ .

Area of a parallelogram:

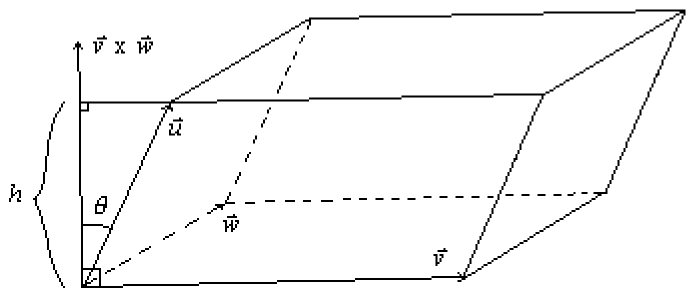



Area of a parallelogram = base times height

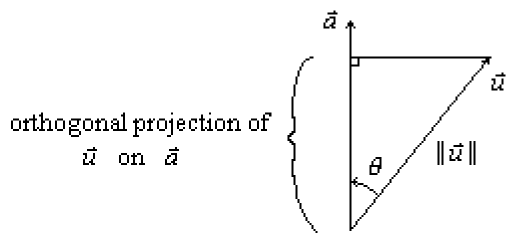
$$\begin{aligned} &= \|\vec{u}\|\left(\|\vec{v}\|\left(\sin(\theta)\right)\right)\ u^2 \\ &= \|\vec{u}\|\|\vec{v}\|\left(\sin(\theta)\right)\ u^2 \\ &= \|\vec{u} \times \vec{v}\|\ u^2 \end{aligned}$$

, from previous work.

Parallelepiped




$h$ is the height of the parallelepiped, which is the orthogonal projection of $\vec{u}$ on the cross product of $\vec{v}$ and $\vec{w}$ .




$$\cos(\theta) = \frac{\text{orthogonal projection of } \vec{u} \text{ on } \vec{a}}{\|\vec{u}\|}$$

The dot product of $\vec{u}$ and $\vec{a}$ ,

$$\vec{u} \bullet \vec{a} = \|\vec{u}\|\|\vec{a}\|(\cos(\theta)) .$$

So,

$$\cos(\theta) = \frac{\vec{u} \bullet \vec{a}}{\|\vec{u}\|\|\vec{a}\|}$$

$$\|\vec{u}\|(\cos(\theta)) = \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|}$$

From before, orthogonal projection of $\vec{u}$ on $\vec{a}$

$$=\|\vec{u}\|(\cos(\theta))$$
$$=\frac{|\vec{u}\bullet\vec{a}|}{\|\vec{a}\|}$$

where the absolute of the dot product between $\vec{u}$ and $\vec{a}$ locates the orthogonal projection of $\vec{u}$ on $\vec{a}$ in terms of the angle $\theta$ (just in case $\frac{\pi}{2} < \theta < \pi$ ).

$h$ of the parallelepiped:

$h = \|$ orthogonal projection of $\vec{u}$ on $\vec{v} \times \vec{w} \|$ , $\vec{v} \times \vec{w}$ perpendicular to the plane of $\vec{v}$ and $\vec{w}$ .

$=\frac{|\vec{u}\bullet(\vec{v}\times\vec{w})|}{\|\vec{v}\times\vec{w}\|} \quad u^1$ , the absolute value of

the dot product of $\vec{u}$ and $(\vec{v}\times\vec{w})$ divided by the magnitude of $\vec{v} \times \vec{w}$ .

The volume of the parallelepiped is the area of the base times the height:

$$\text{V of parallelepiped } =\|\vec{v}\times\vec{w}\|u^2\,\frac{|\vec{u}\bullet(\vec{v}\times\vec{w})|}{\|\vec{v}\times\vec{w}\|}u^1$$
$$=|\vec{u}\bullet(\vec{v}\times\vec{w})| \quad u^3$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\vec{u}\bullet(\vec{v}\times\vec{w})=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\text{V of parallelepiped } = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} u^3 = \left|\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}\right| \quad u^3$$

$$= \left| \begin{bmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{bmatrix} \right| \quad u^3$$

$$= \sqrt{\left| \begin{bmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{bmatrix}^2 \right|} \quad u^3$$

$$= \sqrt{\left| \begin{bmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{bmatrix}^T \right|} \quad u^3, \qquad T, \text{ transpose.}$$

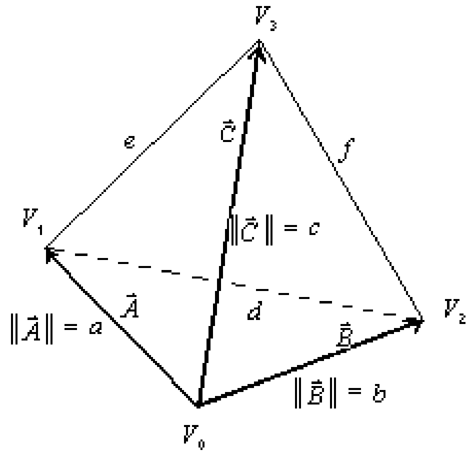
$$= \sqrt{\left| \begin{bmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{bmatrix} \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \right|} \quad u^3$$

$$= \sqrt{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{v} & \vec{u} \bullet \vec{w} \\ \vec{v} \bullet \vec{u} & \vec{v} \bullet \vec{v} & \vec{v} \bullet \vec{w} \\ \vec{w} \bullet \vec{u} & \vec{w} \bullet \vec{v} & \vec{w} \bullet \vec{w} \end{vmatrix}} \quad u^3$$

$$= \sqrt{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{v} & \vec{u} \bullet \vec{w} \\ \vec{u} \bullet \vec{v} & \vec{v} \bullet \vec{v} & \vec{v} \bullet \vec{w} \\ \vec{u} \bullet \vec{w} & \vec{v} \bullet \vec{w} & \vec{w} \bullet \vec{w} \end{vmatrix}} \quad u^3$$

The base of a tetrahedron is a trigon. So,the area of the base of a tetrahedron is $\frac{1}{2}$ the area of the parallelogram base and the volume of a tetrahedron is $\frac{1}{3}$ the product of the area of the base of the tetrahedron times the height of the parallelepiped. That is, it takes $6$ tetrahedrons of the same volume to tessellate a parallelepiped, or the volume of a tetrahedron is $\frac{1}{6}$ the volume of a parallelepiped.

$$m(c_3) = \frac{1}{6}\sqrt{\begin{vmatrix} \vec{u}\bullet\vec{u} & \vec{u}\bullet\vec{v} & \vec{u}\bullet\vec{w} \\ \vec{u}\bullet\vec{v} & \vec{v}\bullet\vec{v} & \vec{v}\bullet\vec{w} \\ \vec{u}\bullet\vec{w} & \vec{v}\bullet\vec{w} & \vec{w}\bullet\vec{w} \end{vmatrix}} \quad u^3$$




$$m(c_3) = \frac{1}{6}\sqrt{\begin{vmatrix} \vec{A}\bullet\vec{A} & \vec{A}\bullet\vec{B} & \vec{A}\bullet\vec{C} \\ \vec{A}\bullet\vec{B} & \vec{B}\bullet\vec{B} & \vec{B}\bullet\vec{C} \\ \vec{A}\bullet\vec{C} & \vec{B}\bullet\vec{C} & \vec{C}\bullet\vec{C} \end{vmatrix}} \quad u^3$$

$$\vec{A}\bullet\vec{A} = \|\vec{A}\|\|\vec{A}\|(\cos(0rad))$$

$$= \quad a^2$$

$$\vec{A} \bullet \vec{B} = \frac{a^2 + b^2 - d^2}{2}$$ (law of cosines)

$$\vec{A} \bullet \vec{C} = \frac{a^2 + b^2 - e^2}{2}$$ (law of cosines)

$$\vec{B}\bullet\vec{B} = \|\vec{B}\|\|\vec{B}\|(\cos(0rad))$$

$$= \quad b^2$$

$$\vec{B} \bullet \vec{C} = \frac{b^2 + c^2 - f^2}{2}$$ (law of cosines)

$$\vec{C}\bullet\vec{C} = \|\vec{C}\|\|\vec{C}\|(\cos(0rad))$$

$$= \quad c^2$$

$$m(c_3)=\frac{1}{3!}\sqrt{\begin{vmatrix} a^2 & \frac{a^2+b^2-d^2}{2} & \frac{a^2+c^2-e^2}{2} \\ \frac{a^2+b^2-d^2}{2} & b^2 & \frac{b^2+c^2-f^2}{2} \\ \frac{a^2+c^2-e^2}{2} & \frac{b^2+c^2-f^2}{2} & c^2 \end{vmatrix}}\quad u^3$$

$$=\frac{1}{3!}\sqrt{\frac{1}{2^3}\begin{vmatrix} 2a^2 & 2\left(\frac{a^2+b^2-d^2}{2}\right) & 2\left(\frac{a^2+c^2-e^2}{2}\right) \\ 2\left(\frac{a^2+b^2-d^2}{2}\right) & 2b^2 & 2\left(\frac{b^2+c^2-f^2}{2}\right) \\ 2\left(\frac{a^2+c^2-e^2}{2}\right) & 2\left(\frac{b^2+c^2-f^2}{2}\right) & 2c^2 \end{vmatrix}}\quad u^3$$

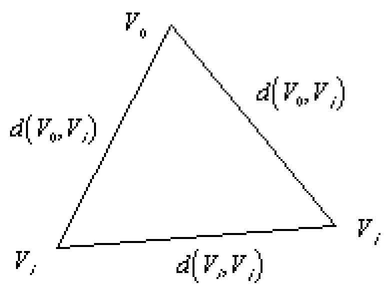
$$=\frac{1}{3\,!}\sqrt{\frac{1}{2^3}\begin{vmatrix} 2a^2 & a^2+b^2-d^2 & a^2+c^2-e^2 \\ a^2+b^2-d^2 & 2b^2 & b^2+c^2-f^2 \\ a^2+c^2-e^2 & b^2+c^2-f^2 & 2c^2 \end{vmatrix}}\quad u^3$$

$$=\frac{1}{3\,!}\sqrt{\frac{1}{2^3}\;D_{3x3}}\quad u^3$$

where

$$\begin{aligned}
a_{11} &= 2a^2 \\
&= a^2 + a^2 - 0^2 \\
&= \left(num\left(d\left(V_{0,}V_1\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_1\right)\right)\right)^2 - \left(num\left(d\left(V_{1,}V_1\right)\right)\right)^2 \\
a_{12} &= a^2 + b^2 - d^2 \\
&= \left(num\left(d\left(V_{0,}V_1\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_2\right)\right)\right)^2 - \left(num\left(d\left(V_{1,}V_2\right)\right)\right)^2 \\
&= a_{21} \\
a_{13} &= a^2 + c^2 - e^2 \\
&= \left(num\left(d\left(V_{0,}V_1\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_3\right)\right)\right)^2 - \left(num\left(d\left(V_{1,}V_3\right)\right)\right)^2 \\
&= a_{31} \\
a_{22} &= 2b^2 \\
&= b^2 + b^2 - 0^2 \\
&= \left(num\left(d\left(V_{0,}V_2\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_2\right)\right)\right)^2 - \left(num\left(d\left(V_{2,}V_2\right)\right)\right)^2
\end{aligned}$$

$$a_{23} = b^2 + c^2 - f^2$$
$$= (num(d(V_{0,}V_2)))^2 + (num(d(V_{0,}V_3)))^2 - (num(d(V_{2,}V_3)))^2$$
$$= a_{32}$$
$$a_{33} = 2c^2$$
$$= c^2 + c^2 - 0^2$$
$$= (num(d(V_{0,}V_3)))^2 + (num(d(V_{0,}V_3)))^2 - (num(d(V_{3,}V_3)))^2$$




$$a_{ij} = (num(d(V_{0,}V_i)))^2 + (num(d(V_{0,}V_j)))^2 - (num(d(V_{i,}V_j)))^2$$

**Measures of "Parallelepipeds",** $p_u$ , and constructs, $c_n$ may shed some insight into understanding the formula for the measure of a 4 - D construct.

A 0 - D parallelepiped, $p_0$ , is a point. The number of 0 - D cubes, $u^0$ , also a point, that tessellate $p_0$ is one, 1 . That is, $m(p_0) = 1\ u^0$ . And, one 0 - D construct, $c_0$ , also a point, tessellates $p_0$ .

$$m(c_0) = \frac{1}{1} m(p_0) = \frac{1}{1} 1 \quad u^0$$

$$= \frac{1}{0\ !} \sqrt{\frac{1}{2^0} D_{0x0}} \quad u^0 \text{ , where}$$

$$0\ ! = 1 \text{ , } 2^0 = 1 \text{ , and } D_{0x0} = 1 \text{ .}$$

$$\bullet \qquad p_0 \text{ , } u^0 \text{ , } c_0$$

**Diagram 11.** A diagram of a 0 - D parallelepiped, cube, construct.

The absolute value of the determinant of the matrix of vector coordinates, such that vectors have a common vertex, the origin, and determine the parallelepiped, is the measure of the parallelepiped.

A $p_1$ is determined by a vector, $a$, with an initial vertex, $V_0$, all in a 1 - D space. A 1 - D parallelepiped is a line segment.

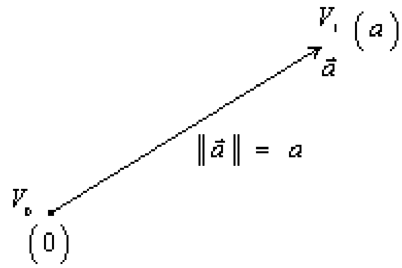



**Diagram 12.** Diagram of a 1 - D parallelepiped, construct, one coordinate system.

$$m(p_1) = |[a]| \quad u^1$$
$$= \sqrt{\left|[a]^2\right|} \quad u^1$$
$$= \sqrt{\left|[a][a]^T\right|} \; u^1$$
$$= \sqrt{|[a \bullet a]|} \quad u^1$$
$$= \sqrt{\left|a^2\right|} \quad u^1$$
$$= \sqrt{\frac{1}{2^1} \left|2a^2\right|} \quad u^1$$

Since one $c_1$ tessellates one $p_1$,

$$m(c_1) = \frac{1}{1}\sqrt{\frac{1}{2^1}\left|2a^2\right|} \quad u^1$$
$$= \frac{1}{1\ !} \sqrt{\frac{1}{2^1} \left|2a^2\right|} \quad u^1$$
$$= \frac{1}{1\ !} \sqrt{\frac{1}{2^1} D_{1x1}} \quad u^1$$
$$a_{11} = (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_1)))^2 - (num(d(V_{1,}V_1)))^2$$
$$= \quad 2a^2$$

A $p_2$, is determined by two vectors, $a$, $b$ with a common vertex, $V_0$, in a 2 - D space.

A 2 - D parallelepiped is a parallelogram.

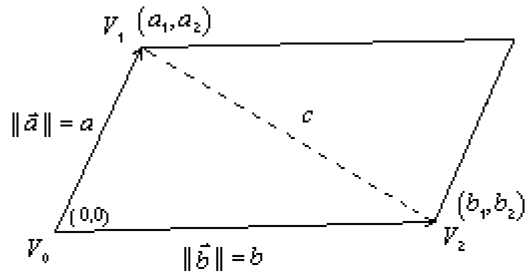



**Diagram 12.** Diagram of a 2 - D parallelepiped, a parallelogram, two coordinate system.

$$m(p_2) = \left\| \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \right\| u^2$$

$$= \sqrt{\left| \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}^2 \right|} \; u^2$$

$$= \sqrt{\left| \begin{bmatrix} a \\ b \end{bmatrix}^2 \right|} \; u^2$$

$$= \sqrt{\left| \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}^T \right|} \; u^2 \qquad = \sqrt{\left| \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} \right|} \; u^2$$

$$m(p_2) = \sqrt{\begin{vmatrix} a \bullet a & a \bullet b \\ b \bullet a & b \bullet b \end{vmatrix}} \; u^2$$

$$a \bullet a = a^2$$

$$a \bullet b = b \bullet a = \frac{a^2 + b^2 - c^2}{2}$$ , law of cosines

$$b \bullet b = b^2$$

$$m(p_2) = \sqrt{\begin{vmatrix} a^2 & \dfrac{a^2 + b^2 - c^2}{2} \\ \dfrac{a^2 + b^2 - c^2}{2} & b^2 \end{vmatrix}} \; u^2$$

$$m(p_2)=\sqrt{\frac{1}{2^2}\begin{vmatrix} 2a^2 & a^2+b^2-c^2 \\ a^2+b^2-c^2 & 2b^2 \end{vmatrix}}\quad u^2$$

Since two mirror-image congruent $c_2$ tessellate one $p_2$ ,

$$m(c_2)=\frac{1}{2}\sqrt{\frac{1}{2^2}\begin{vmatrix} 2a^2 & a^2+b^2-c^2 \\ a^2+b^2-c^2 & 2b^2 \end{vmatrix}}\; u^2$$

$$= \quad \frac{1}{2\;!}\;\sqrt{\frac{1}{2^2}\;D_{2x2}}\quad u^2$$

$$a_{11} = (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_1)))^2 - (num(d(V_{1,}V_1)))^2$$
$$= \quad 2a^2$$
$$a_{12} = (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_2)))^2 - (num(d(V_{1,}V_2)))^2$$
$$= \quad a^2 \;+\; b^2 \;-\; c^2$$

$$= \quad a_{21}$$
$$a_{22} = (num(d(V_{0,}V_2)))^2 + (num(d(V_{0,}V_2)))^2 - (num(d(V_{2,}V_2)))^2$$
$$= \quad 2b^2$$

A $p_3$ is determined by three vectors, $\vec{a}$, $\vec{b}$, $\vec{c}$ with a common vertex, $V_0$, in 3 - D space.

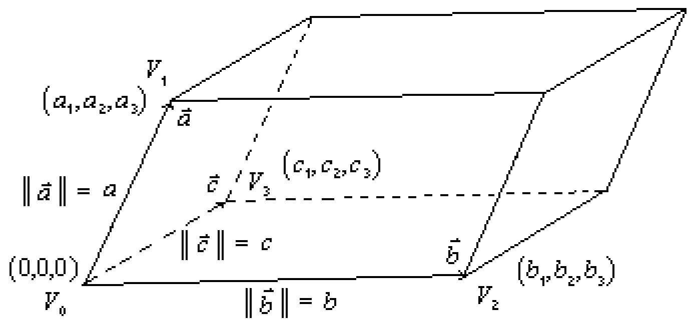



**Diagram 13**, A 2 - D image of a $p_3$ , three coordinate system.

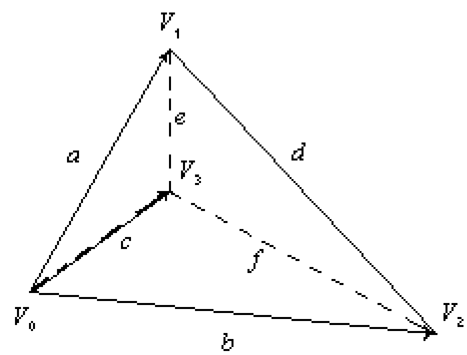



**Diagram 14**. 2 - D image of a $c_3$ such that six $c_3$ with the same measure tessellate $p_3$ .

$$m(p_3) = \left\| \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right\| \quad u^3$$

$$= \sqrt{\left| \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^2 \right|} \quad u^3$$

$$= \sqrt{\left| \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix}^2 \right|} \quad u^3$$

$$= \sqrt{\left| \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix}^T \right|} \; u^3$$

$$= \sqrt{\left| \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix} \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} \right|} \; u^3$$

$$m(p_3) = \sqrt{\begin{vmatrix} \vec{a} \bullet \vec{a} & \vec{a} \bullet \vec{b} & \vec{a} \bullet \vec{c} \\ \vec{b} \bullet \vec{a} & \vec{b} \bullet \vec{b} & \vec{b} \bullet \vec{c} \\ \vec{c} \bullet \vec{a} & \vec{c} \bullet \vec{b} & \vec{c} \bullet \vec{c} \end{vmatrix}} \; u^3$$

$$a_{11} = \vec{a} \bullet \vec{a} = a^2$$

$$a_{12} = \vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a} = \frac{a^2 + b^2 - d^2}{2} \quad \text{, law of cosines}$$

$$= a_{21}$$

$$a_{13} = \vec{a} \bullet \vec{c} = \vec{c} \bullet \vec{a} = \frac{a^2 + b^2 - e^2}{2} \quad \text{, law of cosines}$$

$$= a_{31}$$

$$a_{22} = \vec{b} \bullet \vec{b} = b^2$$

$$a_{23} = \vec{b} \bullet \vec{c} = \vec{c} \bullet \vec{b} = \frac{b^2 + c^2 - f^2}{2} \quad \text{, law of cosines}$$

$$= a_{32}$$

$$a_{33} = \vec{c} \bullet \vec{c} = c^2$$

$$m(p_3) = \sqrt{\begin{vmatrix} a^2 & \frac{a^2+b^2-d^2}{2} & \frac{a^2+c^2-e^2}{2} \\ \frac{a^2+b^2-d^2}{2} & b^2 & \frac{b^2+c^2-f^2}{2} \\ \frac{a^2+c^2-e^2}{2} & \frac{b^2+c^2-f^2}{2} & c^2 \end{vmatrix}} \; u^3$$

$$= \sqrt{\frac{1}{2^3} \begin{vmatrix} 2a^2 & a^2 + b^2 - d^2 & a^2 + c^2 - e^2 \\ a^2 + b^2 - d^2 & 2b^2 & b^2 + c^2 - f^2 \\ a^2 + c^2 - e^2 & b^2 + c^2 - f^2 & 2c^2 \end{vmatrix}} \; u^3$$

Since six $c_3$ having the same volume tessellate one $p_3$,

$$m(c_3)=\frac{1}{6}\sqrt{\frac{1}{2^3}\begin{vmatrix} 2a^2 & a^2+b^2-d^2 & a^2+c^2-e^2 \\ a^2+b^2-d^2 & 2b^2 & b^2+c^2-f^2 \\ a^2+c^2-e^2 & b^2+c^2-f^2 & 2c^2 \end{vmatrix}}\quad u^3$$

$$= \frac{1}{3\ !}\sqrt{\frac{1}{2^3}\ D_{3x3}}\quad u^3$$

$$a_{11} = (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_1)))^2 - (num(d(V_{1,}V_1)))^2$$

$$= 2a^2$$

$$a_{12} = (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_2)))^2 - (num(d(V_{1,}V_2)))^2$$

$$= a^2 + b^2 - d^2$$

$$= a_{21}$$

$$a_{13} = (num(d(V_{0,}V_1)))^2 + (num(d(V_{0,}V_3)))^2 - (num(d(V_{1,}V_3)))^2$$

$$= a^2 + c^2 - e^2$$

$$= a_{31}$$

$$a_{22} = (num(d(V_{0,}V_2)))^2 + (num(d(V_{0,}V_2)))^2 - (num(d(V_{2,}V_2)))^2$$

$$= 2b^2$$

$$a_{23} = (num(d(V_{0,}V_2)))^2 + (num(d(V_{0,}V_3)))^2 - (num(d(V_{2,}V_3)))^2$$

$$= b^2 + c^2 - f^2$$

$$= a_{32}$$

$$a_{33} = (num(d(V_{0,}V_3)))^2 + (num(d(V_{0,}V_3)))^2 - (num(d(V_{3,}V_3)))^2$$

$$= 2c^2$$

A $p_4$ is determined by four vectors, $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ with a common vertex, $V_0$, $(0,0,0,0)$, in a 4 - D space. The end of $\vec{a}$ is a vertex $V_1$ with coordinates of $(a_1, a_2, a_3, a_4)$, $d(V_0,V_1) = a\ \ u$ or $\|\vec{a}\| = a$; the end of $\vec{b}$ is a vertex $V_2$ with coordinates of $(b_1, b_2, b_3, b_4)$, $d(V_0,V_2) = b\ u$ or $\|\vec{b}\| = b$ and $d(V_1,V_2) = e\ u$; the end of $\vec{c}$ is a vertex $V_3$ with coordinates of $(c_1,\ c_2,\ c_3,\ c_4)$, $d(V_0,V_3) = c\ u$ or $\|\vec{c}\| = c$ and $d(V_1,V_3) = f\ \ u$; the end of $\vec{d}$ is a vertex $V_4$ with coordinates of $(d_1, d_2, d_3, d_4)$, $d(V_0,V_4) = d\ u$ or $\|\vec{d}\| = d$ and $d(V_1,V_4) = g\ \ u$, $d(V_2,V_3) = h\ \ u$, $d(V_2,V_4) = i\ \ u$, and $d(V_3,V_4) = j\ \ u$. A 2 - D image of the preceding $p_4$ or $c_4$ is more marginal than the previous 2 - D image of $p_3$ or $c_3$.

$$m(p_4) = \left\| \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} \right\| \quad u^4$$

$$= \sqrt{\left| \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}^2 \right|} \quad u^4$$

$$= \sqrt{\left| \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \\ \vec{d} \end{bmatrix}^2 \right|} \quad u^4$$

$$= \sqrt{\left| \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \\ \vec{d} \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \\ \vec{d} \end{bmatrix}^T \right|} \; u^4$$

$$= \sqrt{\left| \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \\ \vec{d} \end{bmatrix} \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} & \vec{d} \end{bmatrix} \right|} \quad u^4$$

$$m(p_4) = \sqrt{\begin{vmatrix} \vec{a} \bullet \vec{a} & \vec{a} \bullet \vec{b} & \vec{a} \bullet \vec{c} & \vec{a} \bullet \vec{d} \\ \vec{b} \bullet \vec{a} & \vec{b} \bullet \vec{b} & \vec{b} \bullet \vec{c} & \vec{b} \bullet \vec{d} \\ \vec{c} \bullet \vec{a} & \vec{c} \bullet \vec{b} & \vec{c} \bullet \vec{c} & \vec{c} \bullet \vec{d} \\ \vec{d} \bullet \vec{a} & \vec{d} \bullet \vec{b} & \vec{d} \bullet \vec{c} & \vec{d} \bullet \vec{d} \end{vmatrix}} \quad u^4$$

$$a_{11} = \vec{a} \bullet \vec{a} = a^2$$

$$a_{12} = \vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a} = \frac{a^2 + b^2 - e^2}{2} \text{, law of cosines}$$

$$= a_{21}$$

$$a_{13} = \vec{a} \bullet \vec{c} = \vec{c} \bullet \vec{a} = \frac{a^2 + c^2 - f^2}{2} \text{, law of cosines}$$

$$= a_{31}$$

$$a_{14} = \vec{a} \bullet \vec{d} = \vec{d} \bullet \vec{a} = \frac{a^2 + d^2 - g^2}{2}, \text{ law of cosines}$$
$$= a_{41}$$

$$a_{22} = \vec{b} \bullet \vec{b} = b^2$$
$$a_{23} = \vec{b} \bullet \vec{c} = \vec{c} \bullet \vec{b} = \frac{b^2 + c^2 - h^2}{2}, \text{ law of cosines}$$
$$= a_{32}$$
$$a_{24} = \vec{b} \bullet \vec{d} = \vec{d} \bullet \vec{b} = \frac{b^2 + d^2 - i^2}{2}, \text{ law of cosines}$$
$$= a_{42}$$

$$a_{33} = \vec{c} \bullet \vec{c} = c^2$$
$$a_{34} = \vec{c} \bullet \vec{d} = \vec{d} \bullet \vec{c} = \frac{c^2 + d^2 - j^2}{2}, \text{ law of cosines}$$
$$= a_{43}$$

$$a_{44} = \vec{d} \bullet \vec{d} = d^2$$

$$m(p_4) = \sqrt{\begin{vmatrix} a^2 & \frac{a^2+b^2-e^2}{2} & \frac{a^2+c^2-f^2}{2} & \frac{a^2+d^2-g^2}{2} \\ \frac{a^2+b^2-e^2}{2} & b^2 & \frac{b^2+c^2-h^2}{2} & \frac{b^2+d^2-i^2}{2} \\ \frac{a^2+c^2-f^2}{2} & \frac{b^2+c^2-h^2}{2} & c^2 & \frac{c^2+d^2-j^2}{2} \\ \frac{a^2+d^2-g^2}{2} & \frac{b^2+d^2-i^2}{2} & \frac{c^2+d^2-j^2}{2} & d^2 \end{vmatrix}} \; u^4$$

$$= \sqrt{\frac{1}{2^4} \begin{vmatrix} 2a^2 & a^2+b^2-e^2 & a^2+c^2-f^2 & a^2+d^2-g^2 \\ a^2+b^2-e^2 & 2b^2 & b^2+c^2-h^2 & b^2+d^2-i^2 \\ a^2+c^2-f^2 & b^2+c^2-h^2 & 2c^2 & c^2+d^2-j^2 \\ a^2+d^2-g^2 & b^2+d^2-i^2 & c^2+d^2-j^2 & 2d^2 \end{vmatrix}} \; u^4$$

Since 24 $c_4$ with the same measure tessellate one $p_4$,

$$m(c_4)=\frac{1}{24}\sqrt{\frac{1}{2^4}\left\|\begin{bmatrix} 2a^2 & a^2+b^2-e^2 & a^2+c^2-f^2 & a^2+d^2-g^2 \\ a^2+b^2-e^2 & 2b^2 & b^2+c^2-h^2 & b^2+d^2-i^2 \\ a^2+c^2-f^2 & b^2+c^2-h^2 & 2c^2 & c^2+d^2-j^2 \\ a^2+d^2-g^2 & b^2+d^2-i^2 & c^2+d^2-j^2 & 2d^2 \end{bmatrix}\right\|}\quad u^4$$

$$= \frac{1}{4\ !}\sqrt{\frac{1}{2^4}\ D_{4x4}}$$

$$a_{11} = \left(num\left(d\left(V_{0,}V_1\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_1\right)\right)\right)^2 - \left(num\left(d\left(V_{1,}V_1\right)\right)\right)^2$$
$$= 2a^2$$
$$a_{12} = \left(num\left(d\left(V_{0,}V_1\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_2\right)\right)\right)^2 - \left(num\left(d\left(V_{1,}V_2\right)\right)\right)^2$$
$$= a^2 + b^2 - e^2$$
$$= a_{21}$$
$$a_{13} = \left(num\left(d\left(V_{0,}V_1\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_3\right)\right)\right)^2 - \left(num\left(d\left(V_{1,}V_3\right)\right)\right)^2$$
$$= a^2 + c^2 - f^2$$
$$= a_{31}$$
$$a_{14} = \left(num\left(d\left(V_{0,}V_1\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_4\right)\right)\right)^2 - \left(num\left(d\left(V_{1,}V_4\right)\right)\right)^2$$
$$= a^2 + d^2 - g^2$$
$$= a_{41}$$
$$a_{22} = \left(num\left(d\left(V_{0,}V_2\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_2\right)\right)\right)^2 - \left(num\left(d\left(V_{2,}V_2\right)\right)\right)^2$$
$$= 2b^2$$
$$a_{23} = \left(num\left(d\left(V_{0,}V_2\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_3\right)\right)\right)^2 - \left(num\left(d\left(V_{2,}V_3\right)\right)\right)^2$$
$$= b^2 + c^2 - h^2$$
$$= a_{32}$$
$$a_{24} = \left(num\left(d\left(V_{0,}V_2\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_4\right)\right)\right)^2 - \left(num\left(d\left(V_{2,}V_4\right)\right)\right)^2$$
$$= b^2 + d^2 - i^2$$
$$= a_{42}$$
$$a_{33} = \left(num\left(d\left(V_{0,}V_3\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_3\right)\right)\right)^2 - \left(num\left(d\left(V_{3,}V_3\right)\right)\right)^2$$
$$= 2c^2$$
$$a_{34} = \left(num\left(d\left(V_{0,}V_3\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_4\right)\right)\right)^2 - \left(num\left(d\left(V_{3,}V_4\right)\right)\right)^2$$
$$= c^2 + d^2 - j^2$$
$$= a_{43}$$
$$a_{44} = \left(num\left(d\left(V_{0,}V_4\right)\right)\right)^2 + \left(num\left(d\left(V_{0,}V_4\right)\right)\right)^2 - \left(num\left(d\left(V_{4,}V_4\right)\right)\right)^2$$
$$= 2d^2$$

$m(c_n)=\left(\frac{1}{n!}\right)m$ **( n - D parallelepiped)**

The measure of any n - D construct, $m(c_n)$, in terms of the measure of a corresponding n - D parallelepiped is $\left(\frac{1}{n!}\right)m(p_n)$. That is, $m(c_n)=\left(\frac{1}{n!}\right)m(p_n)$. A review of appropriate resources reveals that this relationship is reasonably well known. Discussions about the relationship usually stem from fundamentally related resources; geometrical, vector analysis, combinations of geometry and vector analysis, and multiple integration. Most of these discussions, directly or indirectly, are dependent upon a generalization of Cavalieri's Principle to n - D spaces. The intuitive discussion which follows is similarly based upon Cavalieri's generalization.

The facets of each n - D parallelepiped, $p_n$, are $p_{n-1}$ parallelepipeds where $n \geq 1$. The facets of a $p_1$ are the two vertices, each of which are a $p_0$. The facets of $p_2$ are the four edges, each of which are $p_1$. The subfacets of $p_n$ are parallelepipeds, also. The subfacets of a $p_2$ are the four vertices, each of which is a $p_0$. The facets of a $p_3$ are the six faces, each of which is a $p_2$. The subfacets of $p_3$ are the twelve edges, each of which is a $p_1$. And, the sub(subfacets) of $p_3$ are the eight vertices, each of which is a $p_0$. Similarly, each $p_4$ has eight facets, $p_3$, twenty-four subfacets, $p_2$, thirty-two sub(subfacets), $p_1$, and sixteen sub(sub(subfacets)), $p_0$. That is, each $p_n$, $n \geq 1$ can be described in terms of a number of facets or subfacets, such that each facet or subfacet is a parallelepiped.

Similarly, the facets and subfacets of constructs are also constructs. $c_1$ has two facets which are the vertices and each of these is a $c_0$. $c_2$ has three facets which are the edges, $c_1$, and three subfacets which are the vertices, $c_0$. $c_3$ has four facets which are trigons, $c_2$, six subfacets which are the edges, $c_1$, and four sub(subfacets) which are the vertices, $c_0$. $c_4$ has five facets which are tetrahedrons, $c_3$, ten subfacets which are trigons, $c_2$, ten sub(subfacets) which are edges, $c_1$, and five sub(sub(subfacets)) which are the vertices, $c_0$. Constructs, similar to parallelepipeds, can be described in terms of facets and subfacets.

Applications of Cavalieri's Principle generalized to n - D space, $n \geq 1$, indicates that the measure of an n - D parallelepiped can be expressed in terms of the measure of the base and the measure of the height of the parallelepiped. That is, the measure of $p_n$, $m(p_n)$, is the product of the measure of a base of $p_n$ and the corresponding height of $p_n$. Keep in mind that the base of a $p_n$ is a facet of $p_n$ which is a $p_{n-1}$. And, the height of $p_n$ is the perpendicular distance between the base of $p_n$ and the opposite facet of $p_n$.

Consider the following:

$$
\begin{aligned}
m(p_1) &= m(\text{ base } p_1)\, m(\text{ height } p_1), \text{ the base of } p_1 \text{ is a facet, } p_0\,,\\
&= m(p_0)\, m(\text{ height } p_1),\ p_0 \text{ is a point and the measure of a point is } 1u^0\,,\\
&= 1u^0\, m(\text{ height } p_1),\ \text{ the height of } p_1 \text{ has the same measure as } p_1\,.\\
m(p_2) &= m(\text{ base } p_2)\, m(\text{ height } p_2), \text{ the base of a } p_2 \text{ is a facet of } p_2\text{ ,a } p_1\,,\\
&= m(p_1)\, m(\text{ height } p_2),\\
&= 1u^0\, m(\text{ height } p_1),\, m(\text{ height } p_2).\\
m(p_3) &= m(\text{ base } p_3)\, m(\text{ height } p_3), \text{ the base of a } p_3 \text{ is a facet of } p_3\text{ ,a } p_2\,,\\
&= m(p_2)\, m(\text{ height } p_3),\\
&= 1u^0\, m(\text{ height } p_1),\, m(\text{ height } p_2),\, m(\text{ height } p_3).\\
m(p_4) &= m(\text{ base } p_4)\, m(\text{ height } p_4), \text{ the base of a } p_4 \text{ is a facet of } p_4\text{ ,a } p_3\,,\\
&= m(p_3)\, m(\text{ height } p_4),\\
&= 1u^0\, m(\text{ height } p_1),\, m(\text{ height } p_2),\, m(\text{ height } p_3),\, m(\text{ height } p_4).\\
&.\\
&.\\
&.\\
m(p_n) &= 1u^0\, m(\text{ height } p_1)\ldots m(\text{ height } p_n),\\
&= 1u^0 \prod_{i=1}^{n} m(\text{ height } p_i).
\end{aligned}
$$

A well known result, also associated with the generalization of Cavalieri's Principle, is about the measure of any n - D pyramid. The measure of any n - D pyramid is one *n*th the product of the measure of the base of the pyramid and the measure of the height of the pyramid.

$$m(\text{ n - D pyramid}) = \left(\frac{1}{n}\right) m(\text{ base pyramid})\, m(\text{ height pyramid}).$$

Every construct is the simplest of pyramids. Hence,

$$m(c_n) = \left(\frac{1}{n}\right) m(\text{ base } c_n)\, m(\text{ height } c_n).$$

Consider the following:

$$
\begin{aligned}
m(c_1) &= \left(\frac{1}{1}\right) m(\text{ base } c_1)\, m(\text{ height } c_1), \text{ the base of } c_1 \text{ is a facet, } c_0\,,\\
&= \left(\frac{1}{1}\right) m(c_0)\, m(\text{ height } c_1),\, m(c_0) = 1u^0,\\
&= \left(\frac{1}{1!}\right) 1u^0\, m(\text{ height } c_1), \text{ the height of a } c_1 \text{ has the same measure as } c_1\,.\\
m(c_2) &= \left(\frac{1}{2}\right) m(\text{ base } c_2)\, m(\text{ height } c_2), \text{ the base of a is a facet of } c_2,\ c_1\,,
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{2}\right) m(c_1)\, m(\text{ height } c_2),\\
&= \left(\frac{1}{2}\right)\left(\frac{1}{1!}\right) u^0\, m(\text{ height } c_1),\, m(\text{ height } c_2),\\
&= \left(\frac{1}{2!}\right) u^0\, m(\text{ height } c_1),\, m(\text{ height } c_2).\\
m(c_3) &= \left(\frac{1}{3}\right) m(\text{ base } c_3)\, m(\text{ height } c_3), \text{the base of a } c_3 \text{ is a facet of } c_3, \text{ a } c_2,\\
&= \left(\frac{1}{3}\right) m(c_2)\, m(\text{ height } c_3),\\
&= \left(\frac{1}{3}\right)\left(\frac{1}{2!}\right) u^0\, m(\text{ height } c_1),\, m(\text{ height } c_2),\, m(\text{ height } c_3),\\
&= \left(\frac{1}{3!}\right) u^0\, m(\text{ height } c_1),\, m(\text{ height } c_2),\, m(\text{ height } c_3).\\
m(c_4) &= \left(\frac{1}{4}\right) m(\text{ base } c_4)\, m(\text{ height } c_4), \text{the base of a } c_4 \text{ is a facet of } c_4, \text{a } c_3,\\
&= \left(\frac{1}{4}\right) m(c_3)\, m(\text{ height } c_4),\\
&= \left(\frac{1}{4}\right)\left(\frac{1}{3!}\right) u^0\, m(\text{ height } c_1),\, m(\text{ height } c_2),\, m(\text{ height } c_3),\, m(\text{ height } c_4),\\
&= \left(\frac{1}{4!}\right) u^0\, m(\text{ height } c_1),\, m(\text{ height } c_2),\, m(\text{ height } c_3),\, m(\text{ height } c_4).\\
&\;\;\vdots\\
m(c_n) &= \left(\frac{1}{n!}\right) u^0\, m(\text{ height } c_1) \ldots m(\text{ height } c_n),\\
&= \left(\frac{1}{n!}\right) u^0 \prod_{i=1}^{n} m(\text{ height } c_i).
\end{aligned}
$$

The height of any $p_i$ is a line segment, And, similarly, the height of any $c_i$ is a line segment. If these line segments are correspondingly the same length,
then $\prod_{i=1}^{n} m(\text{ height } p_i) = \prod_{i=1}^{n} m(\text{ height } c_i)$, which is in accordance with applications of Cavalieri's Principle. Alternately, any $c_n$ determines an n - D prism and a pair of mirror image n - D prisms determine an n - D parallelepiped. Consequently,
$m(c_n) = \left(\frac{1}{n!}\right) m(p_n)$.

**The following programs** are designed for use with a TI-82 graphics calculator:

## $c_2$ TRIGON

```
:Disp "THIS PROGRAM"
:Disp "WILL FIND THE"
:Disp "AREA OF ANY"
:Disp "TRIGON:"
:Disp "A<B<C"
:Disp " "
:Disp "ENTER A:"
:Input A
:Disp "ENTER B:"
:Input B
:Disp "ENTER C:"
:Input C
:If A+B>C
:Then
:2A² →G
:A² +B² -C² →H
:2B² →I
:Disp "THE AREA IS:"
:Disp (1/2) √ ((1/4)det [[G,H][H,I]])
:Else
```

## $c_3$ TETRA

```
:Disp "THIS PROGRAM"
:Disp "WILL FIND THE"
:Disp "VOLUME OF ANY"
:Disp "TETRAHEDRON"
:Disp "ENTER A:"
:Input A
:Disp "ENTER B:"
:Input B
:Disp "ENTER C:"
:Input C
:Disp "ENTER D:"
:Input D
:Disp "ENTER E:"
:Input E
:Disp "ENTER F:"
:Input F
:2C² → G
:C² +A² -E² → H
:C² +B² -F² → I
:2A² → J
:A² +B² -D² → K
:2B² → L
:Disp "THE VOLUME IS:"
:Disp (1/6) √ ((1/8)det [[G,H,I][H,J,K][I,K,L]])
```

## $c_4$ PTOPE

```
:Disp "THIS PROGRAM"
:Disp "WILL FIND THE"
:Disp "VOLUME OF ANY"
```

```
:Disp "PENTATOPE"
:Disp "ENTER A:"
:Input A
:Disp "ENTER B:"
:Input B
:Disp "ENTER C:"
:Input C
:Disp "ENTER D:"
:Input D
:Disp "ENTER E:"
:Input E
:Disp "ENTER F:"
:Input F
::Disp "ENTER G:"
:Input G
::Disp "ENTER H:"
:Input H
::Disp "ENTER I:"
:Input I
::Disp "ENTER J:"
:Input J
```

$:2A^2 \rightarrow Q$

$:A^2 + B^2 - E^2 \rightarrow R$

$:A^2 + C^2 - F^2 \rightarrow S$

$:A^2 + D^2 - G^2 \rightarrow T$

$:2B^2 \rightarrow U$

$:B^2 + C^2 - H^2 \rightarrow V$

$:B^2 + D^2 - I^2 \rightarrow W$

$:2C^2 \rightarrow X$

$:C^2 + D^2 - J^2 \rightarrow Y$

$:2D^2 \rightarrow Z$

```
:Disp "THE SPACE"
:Disp "MEASURES"
```

:Disp (1/24) $\sqrt{}$ ((1/16)det [[Q,R,S,T][R,U,V,W][S,V,X,Y][T,W,Y,Z]])

Other programs and a look at the development of the ideas of “real” constructs can be found in a 1995 paper by the authors, “Graphics Calculator Application to Max & Min Problems on Geometric Constructs” posted on the Eighth Annual EPICTCM page of the Math Archives site:

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