Genius of Ramanujan VS. Modern Mathematical Technology

Robert Kreczner
University of Wisconsin-Stevens Point
Stevens Point, WI 54481
Email: rkrczne@uwsp.edu Fax: (715) 346-3624
Telephone: (715) 346-3754

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Abstract. Ramanujan, an Indian mathematician who was labeled as the man who knew infinity, is probably one of the most esoteric mathematical genius in the Twentieth Century. His unexpected, formidable, and at the same time most beautiful formulas ever discovered in Mathematics are a testimony to Ramanujans extraordinary genius. Some of these formulas can be made more accessible, even for most students, by using modern mathematical programs such as Mathematica, Sketch Pad, etc. In this paper, we will consider construction of Ramanujans magic squares, formulas for pi, proving some of his formulas by using symbolic manipulations provided by Mathematica, looking at the problem of partitions.

Contents

1. Introduction

Although most of the Ramanujan's mathematics would be very difficult to understand by most college students, there still can be found several of his problems at the students level. By looking at those "easy problems" we can brink the flavor and expose the students to great mathematics of Ramanujan. The character of Ramanujan's mathematics might be described by the quote of the polish mathematician Mark Kac,see [3], p.281.

The ordinary genius is a fellow that you and I would be just as good as, if we were only many times better. There is no mysteries to how his mind works. Once we understand what he has done, we feel certain that we, too could have done it. It is different with the magicians. They are, to use mathematical jargon , in orthogonal *complement of where we are and the working of their minds is for all intents and purposes incomprehensible. Even after we understand what they have done, the process by which they have done it is completely dark.*

2. Magic Squares

The early work of Ramanujan., probably at the time when he was a school boy, includes construction of different magic squares. As it is generally known a magic square is $n \times n$ table of natural numbers in which a sum along any row, column, or diagonal always equals to the same number. In [1], Chapter1, Ramanujan gives the examples of magic squares of different dimensions, up to 8×8, and suggests some general method to construct them. He also states some of their properties. For example, 3×3 magic square with common sum 60.

28	1	31
23	20	17
9	39	12

In our investigation of magic squares, we will use *Mathematica*[1],mainly its **Solve** function for obtaining solutions of a system of linear equations. First we will look at 2×2 magic squares, the smallest one,

a	b
c	d

with common sum s. By solving corresponding system of equations,

Solve $[\{a{+}b == s, c + d{=}{=}s, a + c = s, b + d == s, a + d == s, c + b == s\}, \{a, b, c, d\}]$

as a *Mathematica* solution we get

$$\left\{\left\{a \to \frac{s}{2}, b \to \frac{s}{2}, c \to \frac{s}{2}, d \to \frac{s}{2}\right\}\right\}. \tag{1}$$

From 1, we see that there is only one type of 2×2 magic squares, mainly he trivial one with all the entries being the same.

For 3×3 magic square

a_{11}	a_{12}	a_{13}
a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}

[1]Version 2.2

the corresponding system of 7 equations with 9 unknowns is

$$\begin{aligned}\mathbf{Solve}[\{a_{11}+a_{12}+a_{13} &= = s, a_{21}+a_{22}+a_{23} == s, a_{31}+a_{32}+a_{33} == s,\\ a_{11}+a_{21}+a_{31} &= = s, a_{12}+a_{22}+a_{32} == s, a_{13}+a_{23}+a_{33} == s\\ a_{11}+a_{22}+a_{33} &= = s, a_{13}+a_{22}+a_{31} == s\},\\ &\quad \{a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}\}]\end{aligned}$$

The solution produced by the *Mathematica*, is

$$\begin{aligned}\{\{a_{32} &\to a_{13}-a_{33}+\frac{s}{3}, a_{12}\to -a_{13}+a_{33}+\frac{s}{3}, a_{21}\to a_{13}+a_{33}-\frac{s}{3},\\ a_{31} &\to -a_{13}+\frac{2s}{3}, a_{32}\to -a_{33}+\frac{2s}{3}, a_{23}\to -a_{13}-a_{33}+s, a_{22}\to \frac{s}{3}\}\}\end{aligned} \tag{2}$$

From the solution 2 we observe that the entries a_{ij} will be integers if s is divisible by 3. Having solution 2, we can now easily compute $a_{22}-a_{21}$and $a_{23-}a_{22}$, which in both cases is the same and equals $-a_{13}-a_{33}+\frac{2s}{3}$, which means that the middle row form an arithmetic sequence. The same observation can be also made about the middle column.

The above observations are contained in the Ramanujan's Notebooks, Part 1, p.17, see [1] ; however, with the assistance of *Mathematica*, those observations are immediate. The construction of larger magic squares with *Mathematica* becomes more cumbersome, but in effect the same, and we will not attempt to demonstrate it here.

3. THE NUMBER PI

In this section we will take look at few Ramanujan's formulas and geometric constructions involving the number π.

One of the simplest approximation formula given by Ramanujan is

$$\pi \approx \sqrt[4]{97\frac{1}{2}-\frac{1}{11}}, \tag{3}$$

which approximate π up to 8 decimal places. The formula 3 can be justified by computing $\mathbf{N}[\pi^4, 8] = 97.40909103$, then replacing it by infinite periodic fraction $97.40909... = 97.4091$, this replacement will cause only small error. Since the last number is closed to 97.5, we compute

$$\mathbf{Rationalize}[97.5-97.4091],$$

which produces the output $\frac{1}{11}$, and thus explaining the reasons for approximation formula 3.

k	0	1	2	3	4	5
$error$	$-7.6\ 10^{-8}$	$-8.8\ 10^{-16}$	$0.\ 10^{-24}$	$0.\ 10^{-32}$	$0.\ 10^{-40}$	$-4.7\ 10^{-48}$

Table 1: Approximation of pi

3.1. Approximation of π by Infinite Series. The series

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26,309n)}{(n!)^4 (396)^{4n}}, \tag{4}$$

discovered by Ramanujan, see [1], Vol. 4, p. 354, was used in 1989 to approximate π up to over one billion of decimal digits. In the Table 1 we display errors for approximating π by the reciprocals of partial sums of that series. To generate the Table 1, we can use *Mathematica* as follows: We define partial sum for 4

$$\mathbf{s}[\mathbf{k\_}] := (\mathbf{Sqrt}[8]/9801)\mathbf{Sum}[((4n)!(1103 + 26390n))/((n!)\hat{}4 * 396\hat{}(4n)), \{n, 0, k\}],$$

then we define the error

$$\mathbf{error}[k\_] := \mathbf{N}[Pi - 1/s[k], 10k],$$

and construct list of errors

$$\mathbf{Table}[\mathbf{errror}[k], \{k, 0, 5\}],$$

for the first six terms.

From the Table 1, we observe that $1/s[k]$ provides exceptionally good approximation of π, each consecutive term of the series 4 improves the approximation by about 8 decimal places. Now we can see why the series 4 has become famous lately , and applied to approximate π up to one billion digits, setting new record.

3.2. Geometric Construction for Approximating π. In [1], Part 3, p.195, there is given a geometrical construction for approximating π. It can be easily made by using *SketchPad*.

The explanation of the construction is given in the Table 2. For all the computations there is needed only the knowledge of Pythagorean Theorem, and properties of similar triangles.

Thus we observe that this construction provides famous $\frac{355}{113}$ approximation of π.

4. PROVING RAMANUJAN'S ALGEBRAIC IDENTITIES BY MATHEMATICA

Most proofs of the incredible Ramanujan's identities are too intricate to be managed directly by *Mathematica*. However, with the set of predefined rules, *Mathematica* can

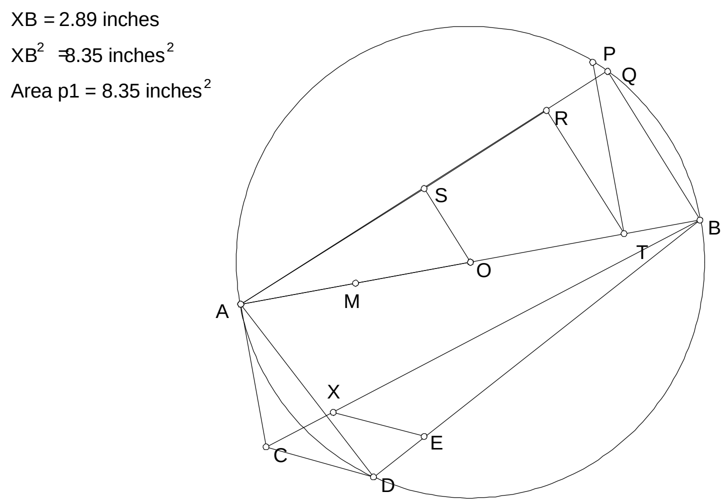

Figure 1: BX approximatelly equals $\sqrt{\pi}$

Construction Steps	**Measurements**
AB is the diameter of the circle	$AB = 2$
Bisect AO at M	$AM = \frac{1}{4}$
Trisect OB at T	$BT = \frac{1}{3}$
Draw TP perpendicular to AB and meeting the circle at P	$PT = \frac{\sqrt{5}}{3}$
Draw $BQ = PT$, and join AQ	$AQ = \frac{\sqrt{31}}{3}$
Draw $OS = PT$, and TR parallel to BQ	$AS = \frac{\sqrt{31}}{6}$, and $SR = \frac{\sqrt{31}}{3}$
Draw $AD = AS$, and $AC = RS$ tangent to the circle at A	$AD = \frac{\sqrt{31}}{6}$, and $AC = \frac{\sqrt{31}}{9}$
Join BC, BD, and CD	$BC = \frac{\sqrt{359}}{9}$, and $BD = \frac{\sqrt{113}}{6}$
Make $BE = BM$	$BE = \frac{3}{2}$
Draw EX parallel to CD	$BX = \sqrt{\frac{355}{115}}$

Table 2: Ramanujan's geometric approximation of pi

assist us to go through those proofs in a relatively painless way. We will illustrate this idea by proving the following Ramanujan's identity.

$$\sum_{k=1}^{n} \frac{1}{n+k} = \frac{n}{2n+1} + \sum_{k=1}^{n} \frac{1}{(2k)^3 - 2k} \tag{5}$$

The **Sum** command, available in *Mathematica*, is not able to prove this identity 5. We will use the following rules:

1. $\sum_{k=1}^{n}(a_k + b_{k)} = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. $\sum_{k=1}^{n} c \cdot a_k = c \sum_{k=1}^{n} a_k$
3. $\sum_{k=1}^{n} a_{n+k} = \sum_{k=1}^{2n} a_k - \sum_{k=1}^{n} a_k$
4. $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_{k-1} + (a_n - a_0)$
5. $\sum_{k=1}^{n} a_{2k-1} = \sum_{k=1}^{2n} a_k - \sum_{k=1}^{n} a_{2k}$

Those four rules can be expressed in *Mathematica* language as follows:
$\mathbf{r1} := \mathbf{Sum}[expr1\_ + expr2\_, \{k\_, 1, n\_\}] -> \mathbf{Sum}[expr1, \{k, 1, n\}] + \mathbf{Sum}[expr2, \{k, 1, n\}]$
$\mathbf{r2} := \mathbf{Sum}[c * expr\_, \{k\_, 1, n\_\}] -> c * \mathbf{Sum}[expr, \{k, 1, n\}]$
$\mathbf{r3} := \mathbf{Sum}[expr\_, \{k\_, 1, n\_\}] :> \mathbf{Sum}[(expr/.n -> 0), \{k, 1, 2n\}] - \mathbf{Sum}[(expr/.n -> 0), \{k, 1, n\}]$
$\mathbf{r4} := \mathbf{Sum}[expr\_, \{k\_, 1, n\_\}] :> \mathbf{Sum}[(expr/.k -> k-1), \{k, 1, 2n\}] + (expr/.k -> n) - (expr/.k -> 0)$
$\mathbf{r5} := \mathbf{Sum}[expr\_, \{k\_, 1, n\_\}] :> \mathbf{Sum}[(expr/.k -> 0.5k + 0.5), \{k, 1, 2n\}] - \mathbf{Sum}(expr/.k -> k - 0.5), \{k, 1, n\}]$

We will show that with these rules the left and right side of formula 5 can be transform to the same expression.

Right Side:

In[1]:=
$\mathbf{Sum}[1/((2k)\hat{}3 - 2k)//\mathbf{Apart}, \{k, 1, n\}] + n/(2n+1)$

Out[1]=
$\frac{n}{1+2n} + \mathbf{Sum}[\frac{-1}{2k} + \frac{1}{2(-1+2k)} + \frac{1}{2(1+2k)}, \{k, 1, n\}]$

In[2]:=
$Out[60]//.\mathbf{r1}$

Out[2]=
$\frac{n}{1+2n} + \mathbf{Sum}[\frac{-1}{2k}, \{k, 1, n\}] + \mathbf{Sum}[\frac{1}{2(-1+2k)}, \{k, 1, n\}] + \mathbf{Sum}[\frac{1}{2(1+2k)}, \{k, 1, n\}]$

In[3]:=

$Out[2]//.\mathbf{r2}$

Out[3]=

$\frac{n}{1+2n} + \frac{\mathbf{Sum}[\frac{-1}{k},\{k,1,n\}]}{2} + \frac{\mathbf{Sum}[\frac{1}{(-1+2k)},\{k,1,n\}]}{2} + \frac{\mathbf{Sum}[\frac{1}{(1+2k)},\{k,1,n\}]}{2}$

In[4]:=

$\mathbf{ReplacePart}[Out[3], Out[3][[4]]/.\mathbf{r4}, 4]//\mathbf{Simplify}$

Out[4]=

$\frac{-\mathbf{Sum}[\frac{1}{k},\{k,1,n\}]}{2} + \mathbf{Sum}[\frac{1}{-1+2k}, \{k,1,n\}]$

Int[5]:=

$\mathbf{ReplacePart}[Out[4], Out[4][[2]]/.\mathbf{r5}, 2]//\mathbf{Symplify}$

Out[5]=

$-\mathbf{Sum}[\frac{1}{2k}, \{, k, 1, n\}] - \frac{\mathbf{Sum}[\frac{1}{k},\{,k,1,n\}]}{2} - \mathbf{Sum}[\frac{1}{k}, \{, k, 1, 2n\}]$

In[6]:=

$Out[5]/.\mathbf{r2}$

Out[6]=

$-\mathbf{Sum}[\frac{1}{k}, \{, k, 1, n\}] + \mathbf{1}.\mathbf{Sum}[\frac{1}{k}, \{, k, 1, 2n\}]$

Left Side

In[7]:=

$\mathbf{Sum}[\frac{1}{n+k}, \{k,1,n\}]/.\mathbf{r3}$

Out[7]=

$-\mathbf{Sum}[\frac{1}{k}, \{, k, 1, n\}] + \mathbf{Sum}[\frac{1}{k}, \{, k, 1, 2n\}]$

Thus, this ends the proof of the identity5.

5. PARTITIONS

The formulas for partitions is, probably, the best know portion of the Ramanujan's work. A partition of a number n is a decomposition of n into a sum of positive integers. Number of all possible partition, disregarding order, for a number n we denote by $p(n)$. Thus, for example,

$$\begin{aligned}
5 &= 1+1+1+1+1 \\
5 &= 1+1+1+2 \\
5 &= 1+1+3 \\
5 &= 1+4 \\
5 &= 2+3 \\
5 &= 2+2+1 \\
5 &= 5
\end{aligned}$$

therefore $p(5) = 7$. Ramanujan discover that $p(n)$ can be asymptotically approximated by the formula

$$p(n) \approx \frac{e^{k\sqrt{n}}}{4n\sqrt{3}}, \tag{6}$$

where $k = \pi\sqrt{\frac{2}{3}}$, and if we approximate the constants in 6, we will get that

$$p(n) \approx 0.144338 \cdot \frac{e^{2.561\sqrt{n}}}{n}$$

By using *Mathematica,* We will try to derive the formula 6 experimentally. We will use the function *Partitions,* which can be accessed by loading *DiscreteMath‘Combinatorica‘* package. In the *Mathematica* we define function $p(n)$,

$In[1] := p[n\_] =$ **Length**[**Partitions**[n]]

The *Mathematica* can evaluate $p[n]$ only for $n \leq 40$, for larger n the list compiled by *Partitions* function will become apparently too large. In this investigation, for the convenience sake, we will consider only $n \leq 30$. We will begin with creating a list of the pairs $(n, Log(np(n))$, and its plot, see the Figure 2. In *Mathematica:*

$t =$ **Table**[$\{n, Log[n * p[n]], \{n, n, 1, 30\}$]

$p =$ **ListPlot**[t]

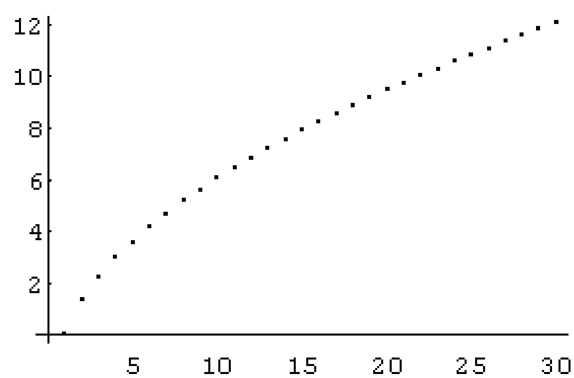

Figure 2: Graph of $\log[np(n)]$

The Figure 2 suggests that the graph of $np(n)$ might be well approximated by a linear combination of the identity and $\sqrt{n}$ functions. To achieve this, we will use the *Fit* function.

$f =$ **Fit**[$t, \{$**Sqrt**[n]$, 1\}, n\}$. Then computing **Exp**[f]//**Expand**, we will get that our approximation of $p(n)$ is given by

$$0.095817\frac{e^{2.63405\sqrt{n}}}{n} \tag{7}$$

n	p(n)	$\mathbf{p}_2(n)$	$\mathbf{p}_1(\mathbf{n})$
1	1	1	2
3	3	3	4
6	11	10	13
9	30	29	35
12	77	73	87
15	176	172	199
18	385	380	427
21	792	797	875
24	1575	1605	1725
27	3010	3121	3285
30	5604	5889	6080

Table 3: Comparison of Ramanujan's formula with our formula

The 3 shows the graph of $p(n)$ and our approximation formula 7 together.

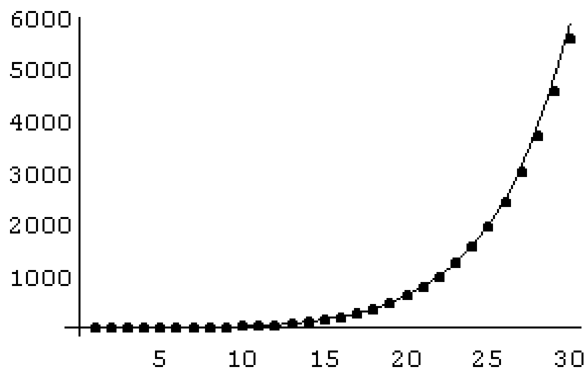

Figure 3: Graphs of List t and Formula 7

The numerical comparison of our formula with Ramanujan's formula is shown in the Table 3, where $p_1(n)$ and $p_2(n)$ is computed by rounding respectively Ramanujan's approximating formula $\frac{e^{k\sqrt{n}}}{4n\sqrt{3}}$, and our approximating formula $0.095817\frac{e^{2.63405\sqrt{n}}}{n}$.

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[1] Bruce C. Bernt, *Ramanujan's Notebooks*, Part1-Part4, Springer-Verlag,1985.

[2] Stephen Wolfram, *Mathematica, A System for Doing Mathematics by Computer,* Second Edition, Addison-Wesley Publishing Company, 1991.

[3] Robert Kanigel, *The Man Who Knew Infinity,* Charles Scribner's Sons, 1991.