Graphics Calculator Applications to Maximum and Minimum Problems on Geometric Constructs

Expressing the size of constructs in terms of the distances between the vertices enables one to easily examine other interesting ideas beyond just the size of a construct (point, line segment, trigon, tetrahedron, pentatope,...). This paper will focus upon the variance, if any, of various attributes of a construct when the distance between a pair of vertices varies. The chosen construct is a 3-D tetrahedron since this can be "really" modelled and allows for discussions involving 3-D, 2-D, 1-D, and 0-D constructs, which can also be readily perceived. Further, to add a little spice to the discussion, 2-D angle and 3-D angle variations will be examined. Although a general formulation for the size of any construct will be exploited, for the purposes of gathering specific data a special tetrahedron has been chosen. This particular tetrahedron is sort of unique in that it is composed of the smallest six consecutive integral lengths that determine a tetrahedron with integral volume. This problem was posed in the Mathematics Magazine, February, 1987, Problem 1261b and solved by this primary author using the prementioned general formula. This general formula was originally formulated and proven for the (0-3)-D constructs by this primary author during November, 1991, and extended to n-D constructs. The extension was proven by Dr. Eugene Curtin and this author during the Fall of 1994. Also, a special 2-D angular ruler, accurate to 10^{-2} radians, was created to directly measure 2-D angles, and L'Huilier's formula was modified to express 3-D angles in steradians. The secondary author formatted the calculators and computers, which greatly enhance response time and cognitive understanding with graphical impact. And, transferred this paper and technological formatings to be published in the electronic proceedings.

The general formula for counting the cubes (point, line segment, square, cube, tesseract,...) of a space that tessellate a construct of that space in terms of the distances between the vertices of the construct is

$$m(C) \;=\; \frac{1}{n\,!}\sqrt{\frac{1}{2^n}\;D_{nxn}}\,u^n$$

where

$$D_{nxn} \;=\; \begin{vmatrix} \alpha_{11} & . \; . \; . & \alpha_{1n} \\ . & . \; . \; . & . \\ \alpha_{n1} & . \; . \; . & \alpha_{nn} \end{vmatrix}, \qquad \alpha_{ij} \quad \text{is}$$

based upon the distances between three vertices, V_o , V_i , V_j as follows and u^n denotes a unit cube of the n-D space that countable tessellates the n-D space.

Choose a vertex, V_o , of the construct. V_i and V_j are any other vertices of the construct.
d(V_o , V_j) represents the distance between vertex V_o and vertex V_j .
n(V_o , V_i) represents the numerical part of that distance between V_o and V_i .

Similarly, $n(V_o, V_j)$ and $n(V_i, V_j)$ represent corresponding numerical parts.

$$\alpha_{ij} = [n(V_0,V_i)]^2 + [n(V_o,V_j)]^2 - [n(V_i,V_j)]^2$$, see the first diagram.
$$= \alpha_{ji}$$

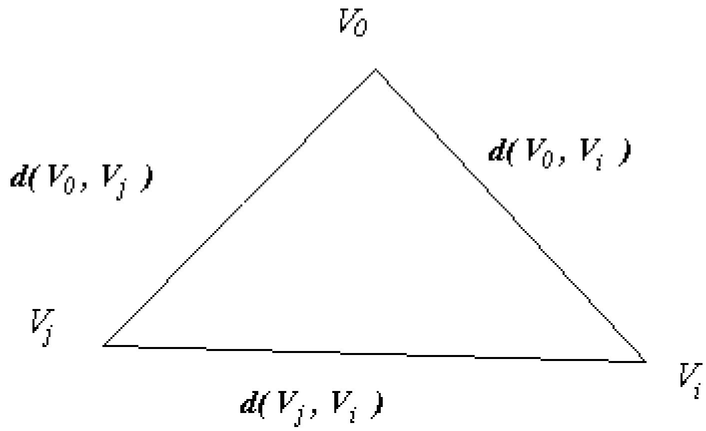

Diagram 1. A representation of the components of α_{ij} .

Special case,

$$\alpha_{kk} = [n(V_0,V_k)]^2 + [n(V_0,V_k)]^2 - [n(V_k,V_k)]^2$$
$$= [n(V_0,V_k)]^2 + [n(V_0,V_k)]^2 - 0$$
$$= 2[n(V_0,V_k)]^2$$, depicted in Diagram 2.

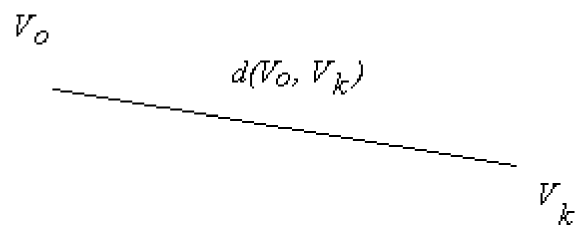

Diagram 2. A representation of the components of special case α_{kk} .

3-D Attributes

The measure of a 3-D construct, the volume of a tetrahedron, as determined by the lengths of the edges is:

$$m\left(V_{(a,b,c,d,e,f)}\right) = \frac{1}{3!}\sqrt{\frac{1}{2^3} D_{3x3}} \quad u^3$$

$$= \frac{1}{6}\sqrt{\frac{1}{8}\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}}\ u^3$$

$$= \frac{1}{6}\sqrt{\frac{1}{8}\begin{vmatrix} 2d^2 & d^2 + e^2 - c^2 & d^2 + f^2 - b^2 \\ d^2 + e^2 - c^2 & 2e^2 & e^2 + f^2 - a^2 \\ d^2 + f^2 - b^2 & e^2 + f^2 - a^2 & 2f^2 \end{vmatrix}}\ u^3,$$

since

$$\alpha_{11} = 2[n(V_0, V_1)]^2 = 2d^2$$
$$\alpha_{12} = \alpha_{21} = [n(V_0, V_1)]^2 + [n(V_0, V_2)]^2 - [n(V_1, V_2)]^2 = d^2 + e^2 - c^2$$
$$\alpha_{13} = \alpha_{31} = [n(V_0, V_1)]^2 + [n(V_0, V_3)]^2 - [n(V_1, V_3)]^2 = d^2 + f^2 - b^2$$
$$\alpha_{22} = 2[n(V_0, V_2)]^2 = 2e^2$$
$$\alpha_{23} = \alpha_{32} = [n(V_0, V_2)]^2 + [n(V_0, V_3)]^2 - [n(V_2, V_3)]^2 = e^2 + f^2 - a^2$$
$$\alpha_{33} = 2[n(V_0, V_3)]^2 = 2f^2$$

as depicted in Diagram 3.

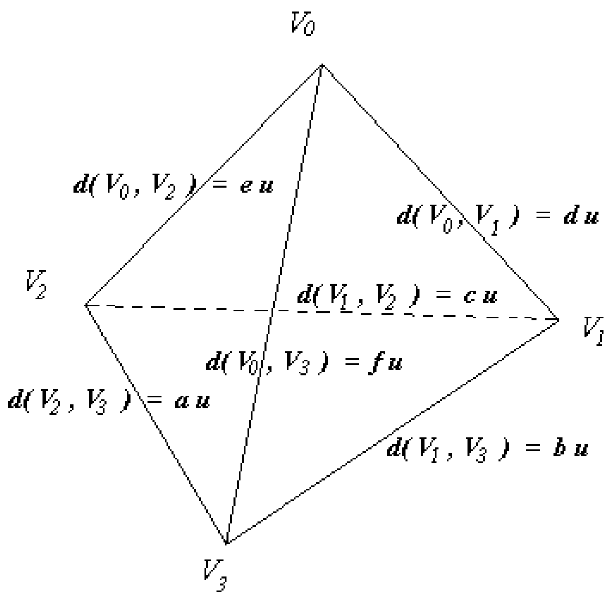

Diagram 3. A general tetrahedron.

To calculate the volume of any tetrahedron, program your calculator to calculate:

$$\frac{1}{6}\sqrt{\frac{1}{8}\begin{vmatrix} 2d^2 & d^2+e^2-c^2 & d^2+f^2-b^2 \\ d^2+e^2-c^2 & 2e^2 & e^2+f^2-a^2 \\ d^2+f^2-b^2 & e^2+f^2-a^2 & 2f^2 \end{vmatrix}}$$

and to ask for "*a,b,c,d,e,f*" as input data. The following TI-82 program accomplishes the calculation of the volume:

```
:Disp "THIS PROGRAM WILL FIND THE VOLUME OF ANY TETRAHEDRON"
:Prompt A,B,C,D,E
:2*D^2→L
:D+E^2-C^2→M
:D^2+F^2-B^2→N
:D^2+E^2-C^2→O
:2*E^2→P
:E^2+F^2-A^2→Q
:D^2+F^2-B^2→R
:E^2+F^2-A^2→S
:2*F^2→T
:(1/6)√((1/8)*det [[L,M,N][O,P,Q][R,S,T]])→V
:Disp "THE VOLUME IS:"
:Disp V
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The tetrahedron chosen as a basis for specific data is the one with the smallest six consecutive integral lengths that determine an integral volume as depicted in Diagram 4:

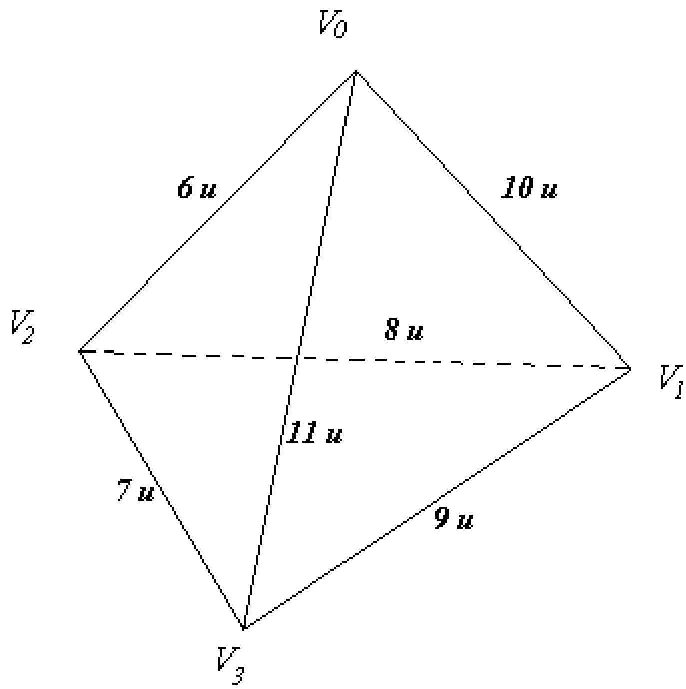

Diagram 4. Tetrahedron with smallest consecutive integral lengths and integral volume.

Check your calculator to determine,

$$m\left(V_{(7,9,8,10,6,11)}\right) = 48u^3$$

For the purposes of gathering data without loss of generality, and discussing variable attributes of this specific trigon, we will allow edge (V_0, V_3) to vary, i.e. f will be identified as a variable, x .

Program a function, f_1 , such that

$$f_1(x) = \frac{1}{6}\sqrt{\frac{1}{8}\begin{vmatrix} 2d^2 & d^2+e^2-c^2 & d^2+x^2-b^2 \\ d^2+e^2-c^2 & 2e^2 & e^2+x^2-a^2 \\ d^2+x^2-b^2 & e^2+x^2-a^2 & 2x^2 \end{vmatrix}}$$

and to ask for "*a,b,c,d,e*" .

The following TI-82 program accomplishes the calculation of the volume as a function of x :

```
:Disp "THIS PROGRAM WILL FIND THE VOLUME OF ANY TETRAHEDRON FOR VALUES OF
        X."
:Prompt A,B,C,D,E
:"(1/6)√((1/8)((2D^2(2E^2*2X^2-(E^2+X^2-A^2)^2))-(D^2+E^2-C^2)((D^2+E^2-C^2)2X^2-(E^2+X^2-A^2)(D^2+X^2-B^2))+(D^2+X^2-B^2)((D^2+E^2-C^2)(E^2+X^2-A^2)-2E^2(D^2+X^2-B^2))))"→Y1
:0→Xmin
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⌛

:18.8→Xmax
:0→Ymin
:60→Ymax
:1→Xscl
:60→Yscl
:DispGraph

The graphics calculator will display the graph of f_1 and can display domain and range values of f_1. The graph combined with ordered pairs allows one to approximate the domain of f_1, i.e. the minimum and maximum values of x and to observe and approximate functional optimal values of $f(x)$, i.e. the minimum volume, 0 u^3, and the maximum volume. The following discrete subsets of f_1 give the functional values of f_1 at integral values of x, subset 1, and a process of refining the minimum and maximum values of x, subset 2, subset 3.

Subset 1 of f_1 describes the volume of the given tetrahedron (to the nearest 10^{-3} u^3) as a function of integral distances between V_0 and V_3.

S_1 = {(3, 17.689), (4, 27.641), (5, 37.777), (6, 42.528), (7, 48), (8, 51.807), (9, 53.599), (10, 52.726), (11, 48), (12, 36.509)}

$f_1(2)$ did not determine a tetrahedron, suggesting that the minimum value for x is between 2 and 3 and, since $f(13)$ did not determine a tetrahedron, the maximum value for x is between 12 and 13. And, $f_1(9)$ suggests a maximum volume of about 53.6 u^3 when the distance is close to 9 u.

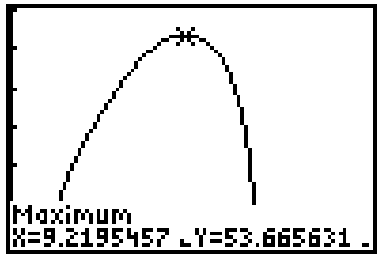
Subset 2 of f_1 similarly describes the volume of the tetrahedron as a function of the distances between V_0 and V_3. Subset 2 is a focus upon approximating the minimum distance as the volume approaches a minimum of 0 $u^{3.}$

S_2 = {(2.2, 4.916), (2.1, not), (2.13, 1.590), (2.12, not), (2.122, 0.309), (2.121, not), (2.1217, 0.064), (2.1216, not), (2.12169, 0.033), (2.12168, not), (2.121687, 0.014), (2.121686, not),...}, <u>not</u> indicating that the value of x did not produce a tetrahedron (discounting degenerate tetrahedrons).

Subset 3 of f_1 describes the volume of the tetrahedron as a function of the distances between V_0 and V_3. Subset 3 is a focus upon approximating the maximum distance as the volume approaches 0 u^3.

S_3 = {(12.8, 10.837), (12.9, not), (12.86, 2.915), (12.87, not), (12.864, 1.068), (12.865, not), (12.8646, 0.192), (12.8647, not), (12.86461, 0.136), (12.86462, not), (12.864619, 0.042), (12.864620, not)}

Graph 1 depicts the approximate minimum x, the approximate maximum x, and the associated volumes as $d(V_0, V_3)$ varies, f_1.



Graph 1. Approximate minimum and maximum distances between the vertices V_0 and V_3 and the aproximate maximum volume of the tetrahedron determined by (7,9,8,10,6,x) associated with f_1.