Examples in Advanced Calculus and Scientific Workplace

Wei-Chi Yang
Department of Mathematics and Statistics
Radford University, Radford, VA 24142
e-mail: wyang@runet.edu

Abstract

We have been talking about the use of computer algebra systems in teaching calculus. I believe we should pay attention to the incorporation of CAS in post calculus courses. In this note I shall demonstrate how easy it is to create teaching materials to enhance students' learining with the help of Scientific Workplace in Advanced Calculus.

1 Introduction

Abstract and complex concepts can be taught and experimented with the help of computer algebra systems. Especially, a difficult course such as Advanced Calculus, teachers usually want students to imagine what a complex plot would look like, which may not be so trivial. In this note, I use the following examples in Advanced Calculus with the help of Scientific Workplace(SWP) to demonstrate how the computer algebra system can be used to enhance students' learning. We note that SWP is not only a wordprocesor but also a computational tool. Thus it can be used in experimenting mathematics without reqiring users to learn Latex and Maple syntax.

2 How do we experiment and construct a function which is nowhere differentiable but everywhere continuous?

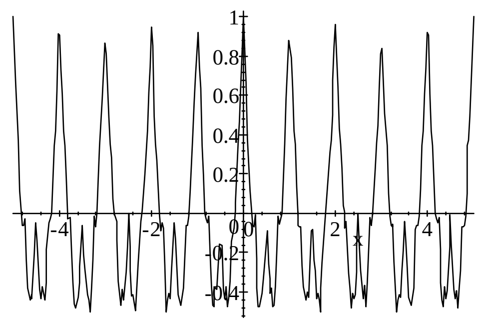
In advanced calculus, we learned that if

$$f(x) = \sum_{k=1}^{\infty} a^k \cos b^k \pi x$$

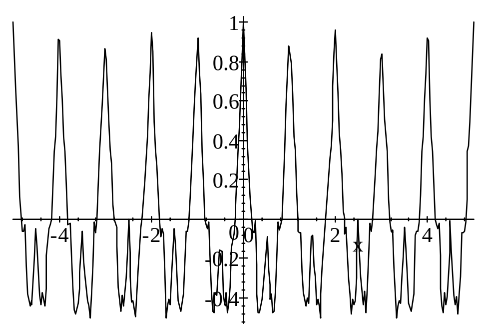
where a and b satisfy certain relationship, then we can prove that the function is nowhere differentiable but continuous everywhere. In this note we shall use the graphing approaches to discover how the behavior of a/b will lead us to a desired nowhere differentiable but continuous function. For a detailed construction, see [1]. We know we can't plot an infinite series of functions but we can use the partial sum to predict the graph of an infinite sum. Thus, first we define the partial sum function as follows:

$$F(a,b,x,n) = \sum_{k=1}^{n} a^k \cos b^k \pi x.$$

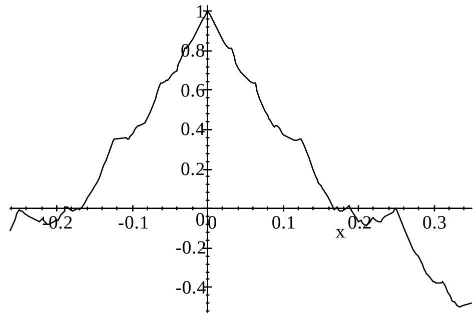
By setting $a = 1/2, b = 2$, and $n = 20$, we graph the function
$F(1/2, 2, x, 20)$



Let's increase the partial sum from $n = 20$ to $n = 30$, and we obtain the following graph.
$F(1/2, 2, x, 30)$

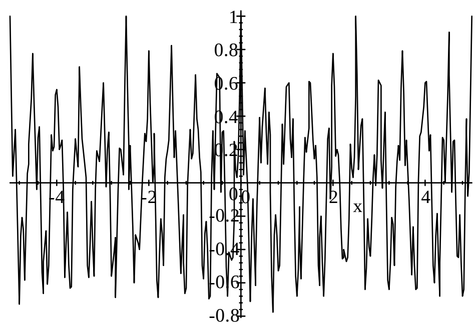


By zooming in many times, we obtain the graph of $F(1/2, 2, x, 30)$ again as follows:



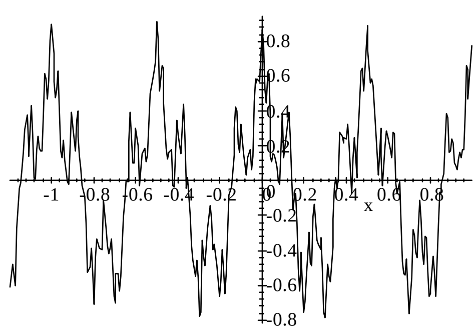
Notice that for $a = 1/2$, and $b = 2$, even if we increase the partial sum, we don't have a function that oscillates as much as we want yet. Therefore, we consider to increase the ratio of $\frac{b}{a}$ from 4 to 8 as follows:

$F(1/2, 4, x, 30)$



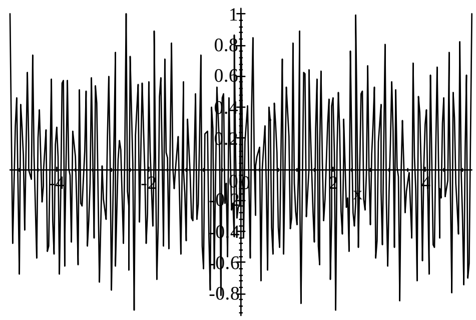
Note that it oscillates more than the previous graph, but still not as much as what we like. We zoom in the graph of $F(1/2, 4, x, 30)$ as follows:

$F(1/2, 4, x, 30)$



We see that the function, $F(1/2, 4, x, 30)$ does have more spikes than that of $F(1/2, 2, x, 30)$. Finally, let's graph $F(1/2, 8, x, 30)$ as follows:

$F(1/2, 16, x, 30)$



This looks like what we want. Therefore, from this worksheet, we learn that to make a highly oscillating trigometric function, such as $\sum_{k=1}^{\infty} a^k \cos b^k \pi x$, to be nowhere differentiable, the key is not to increase its partial sum but to increase the ratio of $\frac{b}{a}$.

3 If the derivative of a function is positive, then the function is increasing?

We see a theorem in advanced calculus like the following:

Theorem 1 *Let f be a function defined on (a, b). Suppose that f is differentiable at $c \in (a, b)$ and $f'(c) > 0$. Then there is a number $\delta > 0$ such that $f(c) < f(x)$ if $x \in (c, c + \delta)$ and $f(x) < f(c)$ if $x \in (c - \delta, c)$.*

We note that it does not say that if $f'(c) > 0$, then there is an interval about c on which f is increasing. For example, let

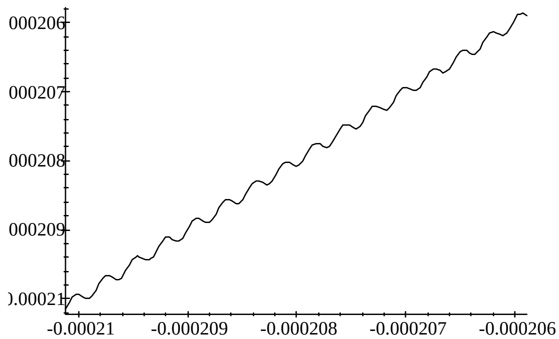
$$f(x) = \begin{cases} x + 2x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then $f'(x) = 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x}$ if $x \neq 0$. (Note that this is computed directly by using **Evaluate** $f'(x)$.). We can see that $f'(0) = 1$ by the following computation:

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = 1.$$

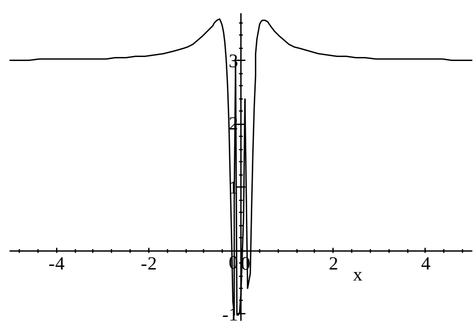
Thus $f'(0) > 0$, it follows from the theorem that there is an interval $(-\delta, \delta)$ so that $f(x) < f(0)$ if $x \in (-\delta, 0)$, and $f(x) > f(0)$ if $x \in (0, \delta)$. In particular, we can take $\delta = \frac{1}{2}$ and see that f is not increasing in the neighborhood of $x = 0$. We can also graph the function f as follows to understand this behavior of f. The following graph of f is obtained by zooming in many times:

f



Notice that the function is oscillating near $x = 0$. This says that the function is not increasing around $x = 0$. We also note that the derivative function is also oscillating near $x = 0$. We graph f' as follows:

$f'(x) = 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x}$



References

[1] Körner, T.W., *Fourier Analysis*, page 38-41, Cambridge University Press, 1988.