A Cupful of Limaçons

`Reference Code  C31`

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Abstract

Have you ever examined the light patterns on the bottom of a cup or bowl with reflective sides and wondered what shapes you see? If a light source is placed somewhat off center, rather than directly above, you might conjecture that the pattern resembles a cardioid. The purpose of this paper is to discuss how students may furnish some answers by using elementary vector algebra and orthogonal projections with the aid of a computer algebra system (CAS). For instance, if the cup is a simple right circular cylinder, then it will become readily apparent that the image is generated by infinitely many limaçons.

We suppose the unit disk $x^2+y^2 \leq 1$ is the bottom of the cup and the sides are formed by revolving the graph of a function $x = f(z)$ with $f(0) = 1$ and $f'(z) \geq 0$ about the z-axis. For example, $f(z) \equiv 1$ when the cup is a cylinder. A CAS such as *Mathematica*, for example, makes it easy to model the reflected light rays formally, without even specifying exactly what $f(z)$ is. By then making particular choices for $f(z)$, we may use the plotting capabilities a CAS provides to assess the accuracy of the model by comparing the graphs to what is seen in reality.

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1. Introduction.

One day a colleague of mine, George Kung, asked if I'd ever contemplated the pattern of reflections on the bottom of a cup. I confessed I hadn't, though a walk outside into the sunlight armed with a coffee mug quickly convinced me that the question was intriguing. If you try such an experiment with the right type of cup (one with reflective sides and bottom works best), you will see a relatively bright image which resembles a cardioid on the bottom of the cup (see Figure 1).

Figure 1. A Cylindrical Cup.

An experiment which we found especially instructive was to sweep out reflection curves by pivoting a laser pointer in such a way so its beam remains fixed at constant heights.

The purpose of this paper is to demonstrate how *Mathematica* may be used by most any third-semester calculus students to investigate and render such images.

2. Reflection Curves.

We begin by choosing the unit disk $\{(x,y,0)\colon x^2+y^2 \le 1\}$ as the base of the cup. If we suppose the sides are formed by revolving the graph of some function $x = f(z)$ with $f(0) = 1$ about the z-axis, then the cross-section of the cup at height z is a circle with radius $f(z)$. For example, $f(z) \equiv 1$ when the cup is a right circular cylinder. Since we intend to use vectors perpendicular to the sides to compute directions of reflected light rays, we will assume f is differentiable with $f' \ge 0$. This nonnegativity assumption on f' prohibits the sides of the cup from narrowing as a function of height, so as to eliminate any complications due to shadowing effects.

If the light source is at infinity, then all incoming light rays can be considered parallel to the xz-plane with the same angle of declination α as shown in Figure 1. Therefore, each light ray has the same direction (i.e., unit vector) given by $\hat{\mathbf{l}} := -\cos\alpha\,\hat{\mathbf{i}} - \sin\alpha\,\hat{\mathbf{k}}$. If we fix any height z, then light rays impact the cup on a circle parametrized by $f(z)\cos\theta\,\hat{\mathbf{i}} + f(z)\sin\theta\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ for $\theta \in [0, 2\pi)$. Since for any fixed θ the vector $f'(z)\cos\theta\,\hat{\mathbf{i}} + f'(z)\sin\theta\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$ is tangent to the cup, we find, after a little bit of thought, that $\vec{\mathbf{n}} := -\cos\theta\,\hat{\mathbf{i}} - \sin\theta\,\hat{\mathbf{j}} + f'(z)\,\hat{\mathbf{k}}$ is a corresponding normal vector. (This construction is similar to that found in the plane, where $a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}}$ and $-b\,\hat{\mathbf{i}} + a\,\hat{\mathbf{j}}$ are orthogonal.) If $\vec{\mathbf{r}}$ denotes a vector parallel to the reflected ray, then we may use orthogonal projections to obtain $\vec{\mathbf{r}} = \hat{\mathbf{l}} - 2\big((\hat{\mathbf{l}}\cdot\vec{\mathbf{n}})/(\vec{\mathbf{n}}\cdot\vec{\mathbf{n}})\big)\vec{\mathbf{n}}$. See the diagram in Figure 2.

Figure 2. Reflection vectors using orthogonal projections.

For each z and θ, a single ray reflects along the line $\vec{\mathbf{R}}(s,\theta) := f(z)\cos\theta\,\hat{\mathbf{i}} + f(z)\sin\theta\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} + s\,\vec{\mathbf{r}}$ where $s \in \mathbb{R}$. We intend to determine the locus of points formed by intersecting these lines with the xy-plane. Some preliminary *Mathematica* definitions are

```
l  :=  {-Cos[a],0,-Sin[a]}
```

```
      n  :=  {-Cos[t],-Sin[t],F'[z]}

R[s_,t_]  :=  {F[z]*Cos[t],F[z]*Sin[t],z}+s*(l-2*l.n/n.n*n)
```

where `a` represents α and `t` represents θ. Simplifying $\vec{\mathbf{R}}(s,\theta)$ via the *Mathematica* command `Simplify[R[s,t],Trig->True]`, one gets

$$\begin{aligned}\vec{\mathbf{R}}(s,\theta) &= \left[f(z)\cos\theta + s\left(-\cos\alpha + \frac{2\cos\theta\,(\cos\alpha\,\cos\theta - f'(z)\sin\alpha)}{1+f'(z)^2}\right)\right]\hat{\mathbf{i}} \\ &\quad + \left[f(z)\sin\theta + \frac{2s\sin\theta\,(\cos\alpha\,\cos\theta - f'(z)\sin\alpha)}{1+f'(z)^2}\right]\hat{\mathbf{j}} \\ &\quad + \left[z + s\left(-\sin\alpha - \frac{2f'(z)\,(\cos\alpha\,\cos\theta - f'(z)\sin\alpha)}{1+f'(z)^2}\right)\right]\hat{\mathbf{k}}.\end{aligned}$$

In order to find the curve of reflection in the xy-plane as a function of θ only, we merely have to set the z-component to zero and solve for s, and then substitute this value back into $\vec{\mathbf{R}}(s,\theta)$ and simplify. These steps are accomplished by using the commands

```
Solve[Simplify[R[s,t][[3]]==0,Trig->True],s]
```

and

```
Simplify[R[%[[1]][[1,2]],t],Trig->True].
```

The symbol `%` is used by *Mathematica* as a synonym for the preceding statement; in this case it was the `Solve` statement. Further simplification, together with a bit of rewriting, results in reflection curve parametrizations

$$\begin{aligned}\vec{\mathbf{R}}(\theta) &:= \left[(A+B\cos\theta)\cos\theta - \frac{z\left(1+f'(z)^2\right)\cos\alpha}{2f'(z)\cos\alpha\,\cos\theta + \left(1-f'(z)^2\right)\sin\alpha}\right]\hat{\mathbf{i}} \\ &\quad + [(A+B\cos\theta)\sin\theta]\,\hat{\mathbf{j}}\end{aligned}$$

where

$$A = \frac{\left(f(z)\left(1 - f'(z)^2\right) - 2zf'(z)\right)\sin\alpha}{2f'(z)\cos\alpha\,\cos\theta + \left(1 - f'(z)^2\right)\sin\alpha}$$

and

$$B = \frac{2\left(f(z)f'(z) + z\right)\cos\alpha}{2f'(z)\cos\alpha\,\cos\theta + \left(1 - f'(z)^2\right)\sin\alpha}.$$

Recall that typical parametric equations for a limaçon are $x(\theta) = r\cos\theta$ and $y(\theta) = r\sin\theta$ where $r = a + b\cos\theta$ and a, b are constants. Closely examining $\vec{\mathbf{R}}(\theta)$, we see that a reflection curve is a horizontal translate of a limaçon iff z satisfies $f'(z) = 0$, in which case $\vec{\mathbf{R}}(\theta) = x(\theta)\,\hat{\mathbf{i}} + y(\theta)\,\hat{\mathbf{j}}$ where $x(\theta) = (A + B\cos\theta)\cos\theta - z\cot\alpha$, $y(\theta) = (A + B\cos\theta)\sin\theta$, $A = f(z)$, and $B = 2z\cot\alpha$. Furthermore, a reflection curve is a translated cardioid precisely when z also satisfies $f(z) = A = B = 2z\cot\alpha$.

In determining the reflection curves we have allowed reflections to take place on either the outside or inside of the cup, and the reflected rays are also allowed to pass through the sides of the cup en route to the xy-plane. Both of these difficulties will be overcome by restricting our attention only to the bottom of the cup when rendering final images.

3. Examples.

Example 1 *(Cylindrical Cup).* If the cup is in the shape of a right circular cylinder, we have $f(z) = 1$ for all $z \geq 0$. After some experimentation, $\alpha = \pi/6 = 30°$ was chosen. Figure 3 below shows the cross-section of the cup as well as the reflection curves on the bottom which correspond to heights $z = 0, 0.02, \ldots, 1.16$. Note that the right-hand portion of Figure 3 is a negative of the actual *Mathematica* plot, so that when this paper is printed white curves and regions on the printout correspond to reflection curves and lighted regions on the bottom of the cup. (The upper limit for z corresponds to $2\tan(\pi/6)$ and represents the maximum allowable height before all reflected rays fail to directly impact the bottom of the cup.) While the boundary of the brighter region superficially resembles a cardioid, we have shown that it is really generated by infinitely many limaçons and one cardioid.

Figure 3. Cross-section and reflection curves for a cylindrical cup.

Example 2 (Cereal Bowl). First, we need to find a function which represents the shape of a given cereal bowl (see the left-hand part of Figure 4). Since the curved portion of the cross-section looks like part of an inverted normal probability function e^{-z^2}, we chose $f(z) = 1 + \sqrt{-\frac{1}{4}\ln(1-z)}$ for $0 \leq z < 1$ to represent the sides of the cereal bowl. (This particular f is the inverse of $1 - e^{-4(z-1)^2}$. There are certainly other functions besides e^{-z^2} which could be suitably modified.) Having found a function which approximates the shape of the bowl, we next need to find a declination angle α which results in a good image on the bottom. Due to the fact that the bowl widens more rapidly than the cylinder of Example 1, we must use a larger angle to guarantee reflected rays impact the bottom. After some trial and error, $\alpha = \pi/2.4 = 75°$ was chosen. Figure 4 contains the results.

Figure 4. Cross-section and reflection curves for a cereal bowl.

It is worth remarking that both the loop and opposing spike are seen in reality.

Figure 1:

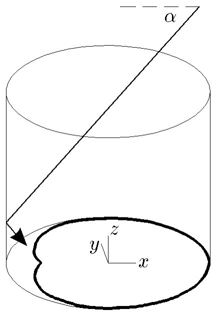

Figure 2:

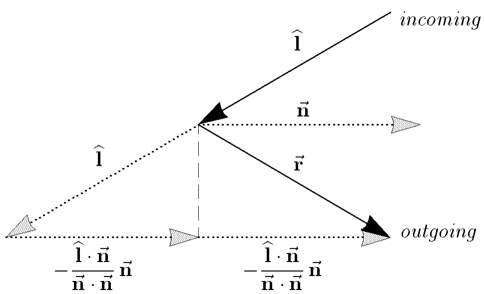

Figure 3:

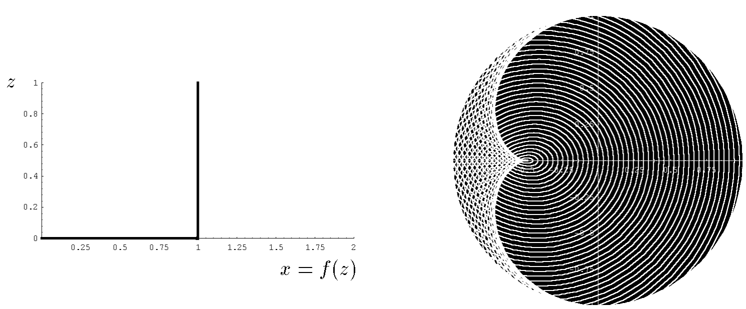

Figure 4: