Explorations in Linear Algebra Using Group Work and Technology

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The focus of much of the reform in collegiate mathematics programs has been Calculus. However, a much quieter reform movement has been underway for the past five years in the field of linear algebra. The Linear Algebra Curriculum Study Group (LACSG) was formed in January 1990 to address the concern that "the linear algebra curriculum at many schools does not adequately address the needs of the students it attempts to serve" (Porter, 1993, p. 41). They found that while the demand for the course from "client disciplines such as engineering, computer science, operations research, economics, and statistics" (Porter, 1993, p. 41) had increased dramatically, then manner and material presented had remained unchanged. Their qualms regarding the emphasis of abstraction of concepts at the expense of real-world applications, the apparent absence of technology used by disciplines that utilize the concepts of linear algebra, and the selection of topics covered have resulted in a restructuring of the typical introductory linear algebra course.

As a result of the recommendations put forth by LACSG the emphasis in linear algebra was shifted to a matrix-oriented course concentrating on applications and reducing the time spent on abstraction of concepts (Porter, 1993, p. 42). While this shift in focus is valuable to both mathematics and non-mathematics majors, the relegation of abstraction to an "also ran" in comparison to applications is doing mathematics majors a great disservice. According to Alan Tucker (1993) "linear algebra was positioned to be the first real mathematics course in the undergraduate mathematics curriculum because its theory is so well structured and comprehensive, yet requires limited mathematical prerequisites" (p. 3). Linear algebra challenges even those undergraduate mathematics majors who succeeded in the first years of calculus. It is the first class where undergraduates are expected to prove theorems and is thus a pivotal course with respect to their ability to conjecture and write coherent proofs. Tucker emphasizes: "A mastery of finite vector spaces, linear transformations, and their extensions to function spaces is essential for a practitioner or researcher in most areas of pure and applied mathematics" (p. 3).

One topic which is being de-emphasized is determinants, specifically, the development and verification of the elementary properties of the determinant (Porter, 1993, p. 43). This shift away from the study of determinants is ironic given the historical development of matrix theory. According to Tucker (1993), determinants (not matrices) developed out of the study of coefficients of systems of linear equations and were used by Leibniz 150 years before the term matrix was coined by J. J. Sylvester in 1848 (p. 5). The crucial link between the newly developed matrix theory and the age old study of determinants was established through the result $\det(AB)=\det(A)\det(B)$ (Tucker, 1993, p. 6). This same result is one of the elementary properties whose development and verification are being eliminated from the curriculum.

The NCTM Curriculum Standards (1989) have woven throughout all levels of education (K-12) the four strands of problem solving, communication, reasoning, and connections. These strands are also echoed in their goals for the student, namely: "(1) that they learn to value mathematics, (2) that they become confident in their ability to do mathematics, (3) that they become mathematical problem solvers, (4) that they learn to communicate mathematically, and (5) that they learn to reason mathematically" (p. 5). They go on to state: "These goals imply that students should be exposed to numerous and varied interrelated experiences that encourage them to value the mathematical enterprise, to develop mathematical habits of mind, and to understand and appreciate the role of mathematics in human affairs; that they should be encouraged to explore, to guess, and even to make and correct errors so that they gain confidence in their ability to solve complex problems; that they should read, write, and discuss mathematics; and that they should conjecture, test, and build arguments about a conjecture's validity" (NCTM, 1989, p. 5). These goals can be reached through the study of linear algebra. However, the content, rich in mathematical explorations, is being weeded out of the curriculum in favor of an applications-based approach. Explorations of properties which connect concepts in linear algebra can enhance the mathematical experience and maturity of those enrolled in the course.

Surprisingly, the use of technology at the collegiate level has been slow to catch on. However, research shows it improves both student achievement and attitudes. Peck et. al. (1994) found that student achievement not only significantly improved in a course which utilized technology, but also in subsequent courses which did not utilize technology. They found that the use of technology "allowed the students to develop their mathematical skills by freeing them to focus on understanding the problems and doing mathematics" (Peck, 1994, p.6). In a study by Quesada and Maxwell (1994), the effects of using graphing calculators to teach pre-calculus were examined. They concluded that the use of graphing calculators improved the achievement of students when compared to students in a traditional course using scientific calculators. The students of the experimental group responded on a survey that they were allowed more exploration, understood the concepts better, and spent more time studying.

The studies performed by Peck et al. and Quesada and Maxwell, as well as those by Guckin and Morrison (1991) and Stiff et al. (1992), demonstrate clearly that students respond with a higher level of achievement and an increase in positive attitudes when they are taught using technology. All of these researchers, however, recognize that the use of technology is the factor that allows them (1) to incorporate real world applications which provide context for topics and (2) to teach their students in a more conceptual, constructivist manner.

Linear algebra is a relative newcomer to the undergraduate mathematics curriculum when compared to the 200 year history of teaching calculus. This does not, however, diminish its significance in a mathematics program. In fact the need for linear algebra as a service course -- a role frequently played by Calculus -- for other degree programs is increasing at a fast pace. Applications for the techniques learned in linear algebra are found in fields as diverse as engineering, physical science, social science, economics and archaeology just to name a few. The influx of students from other degree programs into linear algebra has prompted many instructors to concentrate more on the utility and applications of linear algebra at the expense of removing important abstractions of concepts. Also, other departments choosing to teach their own version of linear algebra has caused severe dilution in abstraction. In response to "turf protection," mathematics departments often choose to water down their curriculum and give "them what they want," just to maintain student enrollment.

The relegation of theory to being a minor player in the development of linear algebra is unfortunate to say the least. linear algebra is a pivotal course in an undergraduate mathematics program. it is the first course where many of the students are challenged as mathematicians. Linear algebra serves as a basis for future mathematical work in group and ring theory, combinatorics, and analysis. Thus students exiting linear algebra need a strong mathematical background and understanding of the concepts involved as a foundation on which the student will build later mathematical knowledge.

The struggle between the utility and beauty of mathematics being the focal point of any mathematics course is a conundrum the instructor must solve. Students should be given the opportunity to explore concepts in linear algebra which are both rich in relationships and in applications.

A module which guides students through the development of the concept of determinants (a topic recently de-emphasized in most introductory linear algebra courses) and focuses on making connections between the determinant of a matrix and other key topics in linear algebra such as the gaussian elimination, inverses, and eigenvalues and eigenvector is available to those who would like a copy. (Both e-mail and postal addresses are at the end of this article.) The module contains five units and a brief description of the content of the module follows.

The overall objectives of this curriculum module are to provide students with:

- opportunities to explore, conjecture, and prove conjectures;
- opportunities to interact with their peers;
- examples which illustrate the efficacy of appropriate technology use in the exploration and development of mathematical concept.

<u>Unit 1 - Determinants as a function</u> - This unit introduces students to the determinant as a function which maps a subset of all matrices with real number entries to the set of real numbers. The students must first describe the subset of matrices which are the domain for the function. Given several examples of matrices, they can use a graphing calculator (TI-81, 82, or 85) to find the determinant of each matrix. If a matrix does not have a determinant, then it is not in the domain of the function. Once they describe the domain of the determinant function, they will use the calculator to explore simple examples (e.g. 2 x 2 case). The student will discover the connection between the entries of the matrix and the determinant of the matrix. After they have derived a

"formula" for finding the determinant of a 2 x 2 matrix, the students will explore special types of matrices (e.g. triangular or diagonal) to find a method of calculating the determinants for special matrices.

Unit 2 - The Effects of EROs (Elementary Row Operations) on the Determinant - In the previous unit, the students discovered a simple method for finding the determinant of upper triangular and diagonal matrices. Unit 2 provides a connection between the technique of gaussian elimination and the calculation of determinants. The students will work cooperatively in pairs with two calculators. One student will use a graphing calculator to perform elementary row operations (EROs) on a given matrix to derive the row echelon form of the matrix. The second student will use a calculator to find the determinant of the resulting matrix after each ERO is performed. Once the pair of students has collected sufficient data, they will attempt to conjecture the effects EROs have on the determinant of a matrix.

Unit 3 - An Exploration of the Properties of Matrices - Unit 3 is a series of investigations of the properties of matrices. The first investigation focuses on the transpose of matrices. The second and third investigations explore inverses and determinants. Once the students have completed the three investigations, they are lead through a series of questions which will help them make connections between the determinant and inverse. This connection will allow the students to broadly classify matrices as being singular or non-singular.

Unit 4 - Geometric Interpretation of Eigenvalues and Eigenvectors - Unit 4 is a teacher lead exploration of the effects of the transformations on the plane which result in a geometric interpretation of eigenvalues and eigenvectors. The students use graphing calculators to iterate the effect a transformation has on a single point in the plane. They then graph the direction and magnitude of the transformation on a transparency. Each group of students are given a different point to iterate. Thus when the transparencies are stacked on top of one another, a rough sketch of the effect of the transformation on the plane appears. A computer program can then be used to provide a more complete picture.

This unit will lead into a discussion on eigenvalues and eigenvectors and their relationship to the determinant of a matrix. Students will be given a series of matrices. They will then (1) find the eigenvalues and eigenvectors, (2) draw a sketch of the transformation for given points, and (3) use the computer program to see the overall effect of the transformation matrix on the plane.

Unit 5 - Eigenvalues of Special Matrices - Unit 5 reinforces and extends the concepts presented in unit 4. the students will be given special matrices of which they must find the eigenvalues and eigenvectors.

Explorations of properties of matrices is a natural means for incorporating technology into the curriculum. Instead of viewing the computer or calculator as a means for alleviating computational burdens, the student (and instructor) will begin to use technology as a means for exploring questions which may not have been tenable. By incorporating technology into the exploration of properties of mathematical objects, we are encouraging students to adopt a mathematical way of thinking. This is the ultimate goal of any mathematics course and of the NCTM Curriculum Standards as well.

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