

## SENSITIVITY ANALYSIS IN LINEAR PROGRAMMING USING A CALCULATOR

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Albert Einstein was to have said that things should be made as simple as possible, but not simpler. When it comes to sensitivity analysis in operations research, the plot does thicken. In this paper we attempt to step in the direction of making what is complex in the context of a dynamic classroom setting, simpler, using the handheld programmable computing devices: TI-85 and HP-48G/GX.

We consider the following changes to a linear programming model (now in a prepared state for solution by the Big M method)

$$\begin{aligned} \max z &= c_1x_1 + c_2x_2 + \dots + c_nx_n \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ (*) \quad & \cdot \qquad \qquad \qquad \cdot \qquad \qquad \cdot \\ & \cdot \qquad \qquad \qquad \cdot \qquad \qquad \cdot \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \\ & x_1 \geq 0, \dots, x_n \geq 0 \end{aligned}$$

after it has been transformed from its initial state (\*) to the optimal model:

- Changes in resource availabilities:  $b_i$
- Changes in a column for a nonbasic variable:  $a_{ij}$  and  $c_j$
- Changes in Resource Availabilities:

Resource availabilities or "scarce resources" are represented by the right sides,  $b_i$  of the constraints. Availabilities are quantities which may be subject to change as time goes on. For example, strikes in an industry producing the needed quantities, bad (or extraordinary) crop years, an increase or decrease in governmental regulations are all possible reasons for changing what is available to a manufacturer, small business, agricultural enterprise, to name a few.

Other changes induced by changes in availabilities:

- The new optimal model may change the actual decision by changing which activities are optimal or, in linear programming terms, bringing in new basic variables.

- Changes in availabilities can also result in changing the optimal value,  $z$ .

Questions to explore:

- Are there changes to the right sides which will not result in a change in the basis? That is, will the same activities still be optimal, although their level may be changed?
- Are there changes to the right sides which will result in a change of basis?
- Is there a range of values for the change in one of the  $b_i$ 's for which the (same) basis remains or changes?
- Will any change to a  $b_i$  cause negative right sides, i.e., an infeasible basis in the final model?

Changing the right side of an initial linear programming model in standard form may lead to an infeasible tableau, i.e., negative right sides. Then a new optimal solution with a new set of basic variables may be obtained using the dual simplex algorithm.

It is possible, however, to change the  $b_i$  's without changing the basis of the optimal solution. This is the possibility which we will explore using the calculator.

Consider the following problem:

An office furniture factory makes desks and chairs. It takes 1 lb of nails to make a desk and 1/4 lb of nails to make a chair. Also, it takes 10 board-feet of lumber to make a desk and 4 board-feet of lumber to make a chair. In any one week there are 300 lbs of nails on hand, and 3,400 board-feet of lumber on hand. The net profit on a desk is \$150 and on a chair, \$100. How many desks and chairs should be made to maximize net profit?

Let

$x_1$  represent the number of desks made  
 $x_2$  represent the number of chairs made

The formulation of the problem is:

$$\begin{aligned} \max \quad & z = 150 x_1 + 100 x_2 \\ \text{s.t.} \quad & x_1 + (1/4)x_2 \leq 300 \\ & 10 x_1 + 4 x_2 \leq 3400 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

The initial representation in standard form is:

$$\begin{aligned}
 & \max z \\
 \text{s.t.} & \quad x_1 + (1/4)x_2 + x_3 + 0 = 300 \\
 (**) & \quad 10x_1 + 4x_2 + 0 + x_4 = 3400 \\
 & \quad -150x_1 - 100x_2 + 0 + 0 + z = 0 \\
 & \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0
 \end{aligned}$$

where  $x_3$  and  $x_4$  are slack variables for the first and second constraints.

The final optimal representation is:

$$\begin{aligned}
 & \max z \\
 \text{s.t.} & \quad (3/8)x_1 + 0 + x_3 - (1/16)x_4 = 87.5 \\
 (***) & \quad (5/2)x_1 + x_2 + 0 + (1/4)x_4 = 850 \\
 & \quad 100x_1 + 0 + 0 + 25x_4 + z = 85000 \\
 & \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0
 \end{aligned}$$

The optimal solution is to make 850 chairs and no desks, for a net profit of \$85,000. And,  $x_3$  and  $x_2$  (in order by rows) are basic variables in the optimal representation.

We want to consider changes,  $\Delta b_i$ , to the right side of the initial representation, which would keep  $x_3$  and  $x_2$  basic and then calculate the new values of those variables and the new optimal value,  $z$ , of the linear program.

A simple way to calculate these quantities is to

(1) Form the matrix:

$$B = \begin{bmatrix} [a_{bv1}] & [a_{bv2}] & \dots & [a_{bvm}] \end{bmatrix} = \begin{bmatrix} [1 & 1/4] \\ [0 & 4] \end{bmatrix} \quad (\text{for the problem above})$$

where the column indices are those associated with the basic variables in the final tableau and the a's are the corresponding columns in the initial tableau.

$$\begin{aligned}
 (2) \text{ Find: } B^{-1} &= \begin{bmatrix} [g_{1,1} \dots g_{1,m}] \\ \dots \\ [g_{m,1} \dots g_{m,m}] \end{bmatrix} = \begin{bmatrix} [1 & -0.0625] \\ [0 & 0.25] \end{bmatrix}
 \end{aligned}$$

(3) Find the range for  $\Delta b_i$  which results in a final feasible tableau with the same basic variables, by forming the inequalities

$$\Delta b_i \geq -d_i / g_{ki}, \quad k = 1, \dots, m$$

where  $d_i$  is the  $i$ th element in the column of right sides in the optimal tableau. See [Harvey, p. 138] for a full discussion of range.

- (4) Select a value for  $\Delta b_i$  from the acceptable range in (3).
- (5) Calculate the new values of the basic variables,  $x_{b_i}$ ,  $i = 1, \dots, m$ .
- (6) Calculate the new optimal value,  $z$ , for the linear program.

The following two programs for the TI-85 will find the acceptable range and new optimal values respectively:

- **RANGE:** The range of values for  $\Delta b_i$  resulting in the same basic variables.
- **NEW:** The new values of the basic variables, and the new optimal value of the linear program for a given  $b_i$  and a given  $\Delta b_i$ .

**RANGE**

```

: Disp "ENTER COL. OF [B]-1 "
: Input J
: Disp "ENTER NUMBER OF ROWS"
: Input M
: 1→K
: Lbl A
: If BINV(K,J)≠0
: Then
: (10-8)→BINV(K,J)
: End
: If K<M
: Then
: K+1→K
: Goto A
: End
: Disp "B INVERSE. PRESS FOR RANGE."
: Disp BINV
: Pause
: 1→K
: {M,1}→dim DEL
: For(K,1,M,1)
: -DOPT(K,1)/BINV(K,J)→DEL(K,1)
; End
: Disp "RANGE. PRESS [ENTER] TO PROCEED"
: Disp DEL [READ THIS VECTOR AS FOLLOWS: Δ ≥ (THE #) IF #< 0 AND Δ ≤ (THE #) IF #> 0]
: Pause
: Disp "PRESS [ENTER] TO PROCEED"
: B-1→BINV
: Fill (0,DEL)

```

To use RANGE have matrices B and B<sup>-1</sup> (BINV) stored in the calculator.

**NEW**

```

: Disp "ENTER NUMBER OF ROWS"
: Input M
: {M,1}→dim F
: Disp "ENTER DELTA ROW"
: Input I
: Disp "ENTER DELTA"

```

```

: Input D
: D→F(I,1)
: Borig+F→NEWB
: BINV*NEWB→X
: CO*BINV*NEWB→NEWZ
: Disp "X. PRESS [ENTER] FOR Z"
: Disp X
: Pause
: Disp NEWZ
: Fill (O,X)
: Fill (O,F)

```

To use NEW, have matrices Borig (original basic  $b_i$ ), CO (original basic objective function coefficients), and BINV stored in the calculator.

NOTE: For the problem above: Borig = [[300] [3400]] and CO = [0 100].

The range for delta is given by [[1400] [-3400]], meaning  $-3400 \leq \Delta \leq 1400$  when we change  $b_2$ . For DEL = -1500,  $X = \{x_3, x_2\} = [[181.25][475.00]]$  and  $z = \$47,500$ .

Try varying DEL for positive values of  $b_1$  and  $b_2$ . How far can they go and still affect  $z$ ?

### ● Changes in a Column for a Nonbasic Variable:

Changing the column ( $c_j$ - and  $a_{ij}$ -values) for a nonbasic variable, like desks, i.e.,  $x_1$ , leaves  $B$ ,  $B^{-1}$ , and hence the right-hand side (which is  $B^{-1}b$ ) of the optimal tableau unchanged. See [Winston, chapt. 6]. Thus:

- An infeasible basis cannot result.

However:

- The basis may become suboptimal, resulting therefore in a new optimal solution: a new set of basic variables and a new optimal  $z$ -value.

Explorations:

- Using the HP-48G/GX we explore nonbasic column parameter conditions that change the basis or keep it the same.

We refer back to the standard form (\*\*) of the office furniture factory problem, noting that it takes 1 lb of nails and 10 board-feet of lumber to make a desk. That the case, we found that  $x_1$ , the number of desks to be made, is nonbasic. Thus no desks should be made. But suppose the price at which a desk is sold and/or the amount of nails and/or lumber used to make a desk are changed. Might the optimal solution change?

Let us suppose, for the sake of an example, that the net profit on a desk is raised to \$250, that only 1/2 lb of nails and 9 board-feet of lumber are to be used for each desk. We now compute the new  $c$ -value,  $c_1$ , using the formula [Winston]

$$c_j = C_{BV} B^{-1} a_{j\text{new}} - c_{j\text{new}}$$

where  $C_{BV}$  is a row matrix of  $c$ -values (from the starting form (\*) of the linear program) corresponding to named basic variables ordered by the row number in which they arise in

the optimal tableau. In this example we have

$$C_{BV} = [0 \ 100], \ a_{1new} = \begin{bmatrix} 1/2 \\ 9 \end{bmatrix}, \ c_{1new} = 250, \ B^{-1} \text{ (as above)}$$

from which we obtain the new c-value:  $c_1 = -25$ . Since the latter is negative, the current basis of  $x_3$  and  $x_2$  is no longer optimal. The fact that  $c_1 = -25$  means that each desk now produced increases profit by \$25. Thus,  $x_1$  should enter into the basis. To find the new optimal solution one must solve the linear program obtained from (\*\*\*) by replacing the column numbers 3/8, 5/2, and 100 by -.0625, 2.25, and -25, resp. [One can make use of a program such as LINDO for this purpose.] The first two of the latter numbers are obtained from  $B^{-1}a_{1new}$  [Winston]. The latter number, -25, is computed as above. The new basis becomes  $x_3$  together with  $x_1$ .

In addition to exploring when a basis changes, we explore changes in the column of a nonbasic variable, like  $x_1$ , for which the basis does not change. We note there is no change in the names of the basis variables when, using the formula above, the computed  $c_j$  is non-negative.

The following HP-48G/GX program has been written for the exploratory purposes cited above, namely, to diagnose whether or not a basis will change. The program accommodates matrices of any dimension, i.e., for problems larger than that carried here. As one experiments by changing parameters, appropriate changes to input matrices must occur. These changes can be accomplished on the HP by a) re-typing the matrices in their entirety and re-entering them, b) using the matrix editing routines found on the HP, or c) using various matrix editing programs such as those described in [LaTorre, chapt. 4].

We note that even in this small example there are three parameters,  $c_1$ ,  $a_{11}$ , and  $a_{12}$ , that are allowed to vary, individually or simultaneously, which makes for a sizeable degree of complexity. The reader is invited to vary the latter parameters, record data, and note if the basis changes: a numerical investigation. It is informative in the case of the example in hand to examine the sensitivity issue from the following additional two points of view: graphical and analytical.

The program below written for the HP-48G/GX will determine whether or not a basis becomes suboptimal, i.e., no longer remains a basis, when any combination of parameters  $c_j$  and  $a_{ij}$  from a single column are altered.

- MAIN PROGRAM: {The program is interactive}

```
P.SA: << P1 P2 P3 P4 P5 P6 SWAP P7 SWAP P8 HALT P9 P9 P10 P9 P11 HALT SWAP
      DROP SWAP DROP P12 P13 DUP2 P14 P15 ROLL P16 NEG P17 P18 >>
      {P.SA = PROGRAM (FOR) SENSITIVITY ANALYSIS}
```

- SUBROUTINES:

```
P1: << "ENTER ROW CBV" PROMPT >> {CBV = CBV}
P2: << "ENTER BINV" PROMPT >> {BINV = B-1}
```

**P3:** << "ENTER COL NBAJ" PROMPT >> {NBAJ = NONBASIC VARIABLE  
 COLUMN J WITH ENTRIES  $a_{ij}$ }  
**P4:** << - CBV BINV NBAJ ' CBV\*BINV\*NBAJ ' >>  
**P5:** << OBJ- DROP >>  
**P6:** << "PRICE MUST " >>  
**P7:** << "EXCEED " >>  
**P8:** << "TO CHANGE BASIS" >>  
**P9:** << " ... " >>  
**P10:** << "CURRENT PRICE IS" >>  
**P11:** << " ENTER CJ " >> {CJ =  $c_j$ }  
**P12:** << 6 >>  
**P13:** << PICK >>  
**P14:** << - A B << IF 'A  $\leq$  B' THEN " NO BASIS CHANGE" ELSE " BASIS CHANGES " END >>>>  
**P15:** << 3 >>  
**P16:** << - >>  
**P17:** << " IS THE NEW CJ " >>  
**P18:** << " END " >>

● **TO RUN THE PROGRAM:**

Press P.SA  
 Enter CBV {See CBV above}  
 Press CONT  
 Enter BINV {See BINV above}  
 Press CONT  
 Enter NBAJ {See NBAJ above}  
 Press CONT  
 Message appears: "PRICE MUST EXCEED..."  
 Press CONT  
 Message appears: "CURRENT PRICE IS"  
 Message appears with request: "ENTER CJ"  
 Enter CJ value  
 Press CONT  
 Final message appears informing whether or not basis changes and giving new CJ value

**REFERENCES**

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**LaTorre, D.R., editor, HP-48G/GX for Science and Engineering Mathematics. Saunders College Publishing, 1994 (Orlando, Florida).**  
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