Dimension of a Maximum Volume

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1. INTRODUCTION. With mathematical computer programs becoming every day more powerful and friendlier, we observe their growing influence on the teaching-learning process in the Mathematics. However, the adaptation of these technologies will a have positive impact on education in Mathematics only if we able to find new problems whose level of difficulty and originality is appropriate to the sophisticated level of present-day software used by the students. While solving such problems, with the aid of computers, we must pay a lot of attention to how the students utilize the empirical data obtained from a computer to support their reasoning. In this process we should emphasize a critical attitude towards these results, and thus introduce the students to idea of proof as a living and important part of the entire process involved in solving problems.

In this article we will present an example of such a problem, which is interesting continuation of the article [**1**] presented at the last ICTCM. Let us denote by $\mathbf{vol(B_p^n)}$ the volume of the unit ball in $\mathbb{R}^n$ determined by the l_p – norm, $p \geq 1$. More precisely, the n-dimensional measure of the set

$$\mathbf{B_p^n} = \{(x_1,...,x_n) \in \mathbb{R}^n\colon |x_1|^p + ... + |x_n|^p \leq 1\}.$$

In [**1**] we showed that

$$\mathrm{vol}(B_p^n) = \frac{\left(\frac{2}{p}\,\Gamma(\frac{1}{p})\right)^n}{\Gamma(\frac{n}{p}+1)} \,. \tag{1.1}$$

By using Mathematica, we can very easily sketch the graphs of $\mathrm{vol}(B_p^n)$ for few fixed values of p. In **Figure 1**, we show the Mathematica output for $p = 2$, 3 and 6.

```
vol[n_,p_]:=((2/p) Gamma[1/p])^n/Gamma[n/p+1]
g2=Plot[vol[n,2],{n,1,20},Frame->True];
g3=Plot[vol[n,3],{n,1,40},Frame->True];
g6=Plot[vol[n,6],{n,1,300},Frame->True];

Show[GraphicsArray[{g2,g3,g6}]]
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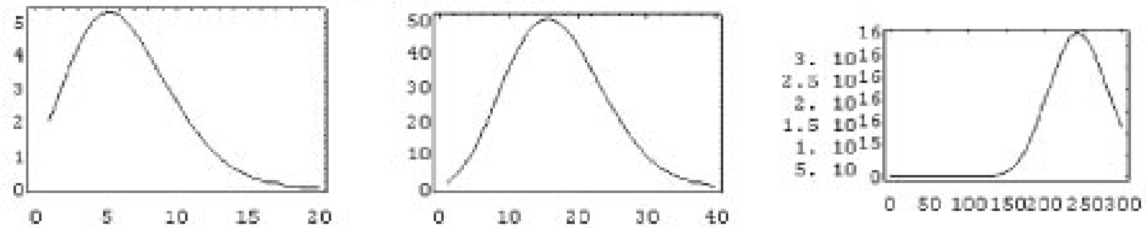

Figure 1. The graphs of vol vs. n, with $p = 1,2$, and 6.

We observe that all these graphs have one hump, which gets higher and moves towards right as p increases. This alone causes some problems when we try to sketch these graphs, because we must at the beginning assign large enough value for n, for each p separately, so that we can see the hole hump. Secondly, there might be more than one hump, but they are simply not displayed by these graphs.

Keeping this dilemma in our mind, we will propose the following problem: **Find $\nu = \nu(p)$, $\nu \in \mathbb{N}$, such that volume of unit balls B_p^n attains its maximum, that is, for fixed $p \geq 1$, $\nu = \nu(p)$ is such a number that**

$$\mathbf{vol}(B_p^n) \leq \mathbf{vol}(B_p^{\nu(p)}), \text{ for all } n \in \mathbb{N}.$$

One of the possible approach to solve this problem would be to apply usual Calculus techniques, the derivatives, but unfortunately this approach will not work here. First, neither the computation of derivative $D_n\mathrm{vol}(B_p^n)$ is possible to carry out for most students, nor is its form obtained from the Mathematica pleasant. Second, trying to solve equation $D_n\mathrm{vol}(B_p^n) = 0$, or even approximate its solution, is harder and messier yet. Third, the solution of this equation would be a real number only approximating $\nu(p)$. Therefore, we must abandon this idea and devise a new approach.

2. SHOWING THAT THERE IS ONLY ONE HUMP. Instead of analyzing the derivative, we will consider quotient **q(n)** given by

$$q(n) = \frac{\mathrm{vol}(B_p^n)}{\mathrm{vol}(B_p^{n-1})},$$

which reminds us little bit of the formula for derivative. With the formula (1.1), and the following basic formulas for Beta (B) and Gamma (Γ) functions:

(i) $\Gamma(\alpha) = \int_0^\infty x^{\alpha\text{-}1}\, e^{\text{-}x}\, dx.$ (iii) $\Gamma(x+1) = x\,\Gamma(x).$

(ii) $B(\alpha, \beta) = \int_0^\infty x^{\alpha\text{-}1}\, (1-x)^{\beta\text{-}1}\, dx.$ (iv) $B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

we will get that

$$q(n) = \frac{2}{p}\int_0^1 (1-x)^{\frac{1}{p}-1}\, x^{\frac{n}{p}}\, dx. \tag{2.1}$$

Formula (2.1) can also be verified by the Mathematica. Since it is easy to observe that

$$(1\text{-}x)^{\frac{1}{p}-1}\ x^{\frac{n+1}{p}} \le (1-x)^{\frac{1}{p}-1}\ x^{\frac{n}{p}},$$

for $0 \le x \le 1$, we conclude that $q(n+1) < q(n)$, that is, the sequence $q(n)$ is decreasing. By analogy to the derivative, we may imagine that the graph of $vol(B_p^n)$, as a discrete set of points, lies on the concave down curve. In addition, $q(0) = 2$, and since the integrand in (2.1) is less than 1, by applying Lebesgue's Theorem, we obtain that $q(n)$ approaches 0 as n approaches positive infinity. From these facts, therefore, we conclude that the function $vol(B_p^n)$ can have only one relative maximum, and this relative maximum must be its absolute maximum at the same time.

3. Let Mathematica Find ν(p). Since we are now confident that the first largest encountered number on the list

$$\{vol(B_p^1),\, vol(B_p^2),...,vol(B_p^n),...\} \tag{3.1}$$

is indeed the absolute maximum, we can determine its position, for few chosen values of p, by employing the Mathematica to do it. For this computation we use little program shown in **Figure 2**, and the results of that computations were placed in the first column of **Table 1**. As we try to compute ν(p), with this program, we discover quickly its two flaws. We must be able to set at front a number k, where k is any

```
In[1]:=
   vol[n_,p_]:=((2/p)Gamma[1/p])^n/Gamma[n/p+1]
In[2]:=


   l[p_,k_]:=Table[N[vol[n,p]],{n,0,k}]
   m[p_,k_]:=Max[l[p,k]]
   n[p_,k_]:=Position[l[p,k],m[p,k],{1}]-1//Flatten
In[5]:=
   nn={n[1,2],n[2,15],n[3,20],n[4,50],n[5,120],n[6,300],
       n[7,600],n[8,1400],n[10,7000]}
Out[5]=
   {{1, 2}, {5}, {16}, {41}, {102}, {242}, {558}, {1263}, {6214}}
```

Figure 2. Mathematica's computation of $\nu(1),\ldots,\nu(10)$

ν	$p\left\lfloor\left(2\,\Gamma\left(\frac{1}{p}+1\right)\right)^p\right\rfloor$	$\nu - p\left\lfloor\left(2\,\Gamma\left(\frac{1}{p}+1\right)\right)^p\right\rfloor$	$\mathrm{mod}_p(\nu(p))$
$\nu(1) = 2$	2	0	0
$\nu(2) = 5$	6	-1	1
$\nu(3) = 16$	15	+1	1
$\nu(4) = 41$	40	+1	1
$\nu(5) = 102$	100	+2	2
$\nu(6) = 242$	240	+2	2
$\nu(7) = 558$	560	-2	2
$\nu(8) = 1263$	1264	-1	1
$\nu(9) = 2817$	2817	0	0
$\nu(10) = 6214$	6210	+4	4

Table 1.

number greater than $\nu(p)$. If k is smaller than $\nu(p)$, list (3.1) is too short, it will not include its absolute maximum, and the obtained output will not be correct. In opposed, if k is too large, the computing time taken by the Mathematica will become too large. As a matter of fact, with the values of k estimated from the graphs of $vol(B_p^n)$, the computation of $\nu(p)$, already for $p > 10$, becomes too long to be useful in practice.

4. SO WHAT ν(p) IS LIKE? Looking at Table 1, first column, we can only observe that $\nu(p)$ becomes rapidly large. In order to comprehend better these large numbers, we sketch the graph of $\ln(\nu(p))$ vs. n, using data from Table 1.

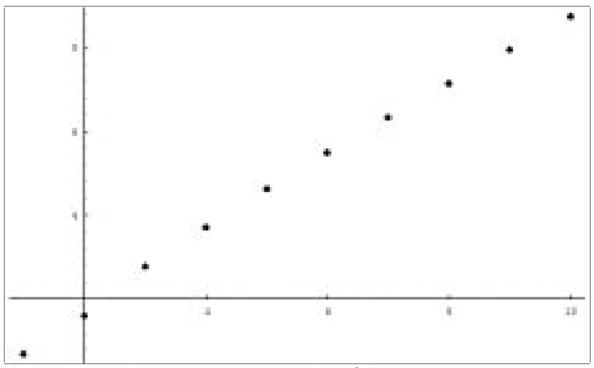

Figure 3. Graph of $\ln(\nu(p))$

Surprisingly, the points lie almost on a straight line, and this indicates almost exponential growth of $\nu(p)$. However, our observation about $\nu(p)$, though impressive, is only a qualitative statement about $\nu(p)$, and it still does not solve our problem to find an exact formula for $\nu(p)$. Since we do not know how to a achieve this goal, we will settle for less, for the time being - we will try to approximate $\nu(p)$.

The main idea, to find a reasonable approximation of $\nu(p)$, is to select a convenient subsequence form sequence $vol(B_p^n)$, and consider the maximum of this subsequence to be an approximation of $\nu(p)$. We decided, for the purpose of simplicity, to select the subsequence $r(n) = vol(B_p^{pn})$, whose formula, by using (1.1), will become

$$vol(B_p^{pn}) = \frac{x^n}{n!}\text{ , where } x = 2^p\,\Gamma(\tfrac{1}{p}+1)^{\,p}\,.$$

Since subsequence r(n) is the every pth term of the sequence vol(B_p^n) and the graph of vol(B_p^n) is of the one $-$ hump shape, the maximum term of the subsequence r(n) lies within p terms to the left or to the right of the maximum term of vol(B_p^n). Further, the nth term in the subsequence r(n) is the $(n \cdot p)$th term in the sequence vol(B_n^p), so an index of the maximum term of the subsequence r(n) multiplied by p should approximate the maximum term of the sequence vol(B_n^p), ν(p), with the error bound $\pm$ p.

On the other hand, we can easily determine the index of the maximum term of r(n), by realizing that the maximum of

$$r(n) = \frac{x^n}{n!} = \frac{x}{1}\,\frac{x}{2}\ldots\frac{x}{n}$$

occurs for the largest n such that $\frac{x}{n} \leq 1$, that is $n \leq x$, that is, $n = \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the floor of x. Thus, we obtained that

$$\nu(p) \approx p \lfloor x \rfloor = p \left\lfloor 2^p\, \Gamma^p(1 + \tfrac{1}{p}) \right\rfloor . \tag{4.1}$$

Now we would like to compare the accuracy of the approximation in (4.1). We already know that the absolute error should be less than p. In **Table 1** this error is displayed in the third column. In the forth column, we computed ν(p) modulo p. The numbers in these two columns are the same except their sign. Accidentally, we discover that

$$\nu(p) = p \left\lfloor 2^p\, \Gamma^p(1 + \tfrac{1}{p}) \right\rfloor \pm \mathrm{mod}_p(\nu(p)). \tag{4.2}$$

Obviously, the impressive formula (4.2) was discovered experimentally, and the the formal proof or disproof of it is still required. However, this seems to be open problem, left to a reader as a challenge to tackle.

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