FAMILIES OF LINEAR FUNCTIONS AND THEIR ENVELOPES

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The topic of envelopes of families of curves was present in calculus texts forty years ago, but has since been deleted (see [1]). Envelopes are discussed in many differential equations texts today, but only as a singular solution to Clairaut's Equation (see [5]). With the advent of Computer Algebra Systems such as Maple-V, Mathematica, and Derive, this topic is accessible to our calculus students and is a good example of the type of problem where the computer can be used as a tool to aid in the solution of a problem. The purpose of this note is to introduce envelopes of families of linear functions, to study some of the geometric properties of envelopes, and finally present a closed form expression for the envelope for a certain class of families of linear functions. We have included a Maple-V worksheet that contains pictures of various families of linear functions. We used these pictures to make conjectures about the corresponding envelope.

Families of Linear Functions

Let $L(\mathbb{R})$ denote the set of linear functions over $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. We define the family of linear functions generated by a differentiable function $f(x)$ to be the set of linear functions of the form $y = ax + f(a)$ for all $a \in \operatorname{dom}(f)$. Our goal is to characterize this family of linear functions. Figure 1 in the Maple-V worksheet shows the family of linear functions generated by the function $f(x) = \sqrt{1 + x^2}$ (the upper branch of the hyperbola $y^2 - x^2 = 1$), i.e. the set of lines of the form $y = ax + \sqrt{1 + a^2}$.

Figure 1 leads us to the conjecture that we may view this family of linear functions as the set of lines tangent to the graph of the upper half of the unit circle. In general we may view the family of linear functions generated by a differentiable function $y = f(x)$ as the set of lines tangent to the graph of a differentiable function $y = g(x)$. The function $g(x)$ is called the envelope of the family of linear functions and our goal is to find the explict form of this function. We use the notation $f(x) \to g(x)$ to denote that the envelope of the family of linear functions $y = ax + f(a)$ is the function $y = g(x)$. Recall that the eqaution of the line tangent to the graph of a differentiable function $y = g(x)$ at the point $(\beta, g(\beta))$ is given by the expression $y = g'(\beta)x + (g(\beta) - \beta g'(\beta))$. This line has the form $y = ax + f(a)$ if and only if $a = g'(\beta)$ and $f(a) = g(\beta) - \beta g'(\beta)$. Substituting $g'(\beta)$ for a in the second equation we see that the function $g(x)$ must satisfy the nonlinear differential equation $g(\beta) - \beta g'(\beta) = f(g'(\beta))$, which is known as Clairaut's equation. If we set m to be the slope and b to be the y-intercept of the line tangent to the graph of $y = g(x)$ at the point $(\beta, g(\beta))$ then we see that m and b must satisfy $b = f(m)$.

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Proposition 1. *If $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1-x^2}$ then $f(x) \to g(x)$.*

Proof. By the discussion above, to show that $f(x) \to g(x)$ we must show that the slope m and the y-intercept b of the line tangent to the graph of $y = g(x)$ satisfy $b = \sqrt{1+m^2}$ (i.e. $b = f(m)$) Since $g(x) = \sqrt{1-x^2}$, then $g'(\beta) = \frac{-\beta}{\sqrt{1-\beta^2}}$ and hence the equation of the line tangent to the graph of $y = g(x)$ at the point $(\beta, g(\beta))$ is given by $y = \frac{-\beta}{\sqrt{1-\beta^2}} x + \frac{1}{\sqrt{1-\beta^2}}$. Thus $m = \frac{-\beta}{\sqrt{1-\beta^2}}$ and $b = \frac{1}{\sqrt{1-\beta^2}}$ and it is clear that

$$f(m) = f\left(\frac{-\beta}{\sqrt{1-\beta^2}}\right) = \sqrt{1 + \left(\frac{-\beta}{\sqrt{1-\beta^2}}\right)^2}$$
$$= \frac{1}{\sqrt{1-\beta^2}} = b.$$

Proposition 1 shows that the envelope of the family of linear functions generated by the upper branch of the hyperbola $y^2 - x^2 = 1$ is the upper half of the unit circle. A straight forward calculation shows that the envelope of the family of linear functions generated by the lower branch of the hyperbola $y^2 - x^2 = 1$ is the lower half of the unit circle. A natural question to ask is whether the envelope of a family of linear functions generated by an arbitrary hyperbola with major axis parallel to the y-axis is a circle or an ellipse.

The Effect that Translations and Dilations of $f(x)$ have on the Envelope $g(x)$

Before answering the question posed above we will define what we mean by translations and dilations of a function $f(x)$. For a more complete disscussion see [2].

(1) The graph of $y = f(x) + E$ is a vertical translation of the graph of $y = f(x)$.
(2) The graph of $y = f(x + C)$ is a horizontal translation of the graph of $y = f(x)$.
(3) The graph of $y = Af(x)$ is a vertical dilation of the graph of $y = f(x)$.
(4) The graph of $y = f(Bx)$ is a horizontal dilation of the graph of $y = f(x)$.

Every hyperbola whose major axis is parallel to the y-axis can be expressed in the form $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$. Solving for y we see that the hyperbola may be described by the two functions $y = \pm\frac{b}{a}\sqrt{a^2 + (x-h)^2} + k$. Setting $f(x) = \sqrt{1+x^2}$ we see that $y = \pm bf(\frac{1}{a}x - \frac{h}{a}) + k$. Thus we see that this hyperbola can be expressed as translations and dilations of $f(x) = \sqrt{1+x^2}$. If we could understand the effect that translations and dilations of $f(x)$ have on the envelope $g(x)$ then we could answer the question posed above. In other words an understanding of the effect that translations and dilations of $f(x)$ have on $g(x)$ would allow us to use a single result ($f(x) \to g(x)$) such as in Proposition 1 to obtain a whole family of related results. In [2] we proved the following result.

Theorem 1. *Suppose $f(x) \to g(x)$, and let $\mathcal{F}(x) = Af(Bx + C) + Dx + E$, and $\mathcal{G}(x) = Ag(\frac{1}{AB}x + \frac{D}{AB}) - \frac{C}{B}x - \frac{CD}{B} + E$ where $A,B,C,D,E \in \mathbb{R}$ and $A,B \neq 0$. Then $\mathcal{F}(x) \to \mathcal{G}(x)$.*

We will now use Proposition 1 and Theorem 1 to find the envelope of the family of linear functions generated by a hyperbola whose major axis is parallel to the y-axis. By

Proposition 1 $f(x) = \sqrt{1+x^2} \to g(x) = \sqrt{1-x^2}$ and by the discussion above the hyperbola may be described by the two functions $\mathcal{F}(x) = \pm bf(\frac{1}{b}x - \frac{h}{a}) + k$. Applying Theorem 1 to $\mathcal{F}(x)$, we see that $\mathcal{F}(x) \to \mathcal{G}(x) = \pm g(\pm\frac{a}{b}x) + hx + k$. Simplifying we see that $\mathcal{G}(x) = \pm\sqrt{b^2 - a^2x^2} + hx + k$. Let $y = \pm\sqrt{b^2 - a^2x^2} + hx + k$, then if we eliminate the radical we see that $(h^2 + a^2)x^2 - 2hxy + y^2 + 2hkx - 2ky + (k^2 - b^2) = 0$. Since $(-2h)^2 - 4(h^2 + a^2) = -4a^2 < 0$ then this conic is a rotated ellipse (see [4]). Thus the envelope of the family of linear functions generated by the hyperbola $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ is indeed an ellipse. Figure 2 in the Maple-V worksheet shows the family of linear functions generated by the hyperbola $(y-2)^2 - \frac{(x-2)^2}{4} = 1$. According to the discussion above, $\mathcal{F}(x) = \pm\frac{1}{2}\sqrt{4 + (x-2)^2} + 2 \to \mathcal{G}(x) = \pm\sqrt{1-4x^2} + 2x + 2$ and the envelope is the ellipse $8x^2 - 4xy + y^2 + 8x - 4y + 3 = 0$.

In [2] we showed more generally that the envelope of a family of linear functions generated by a conic section is again a conic section.

Explicit Form of the Envelope $g(x)$ for a Certain Class of Functions $f(x)$

Given a differentiable function $f(x)$, we saw earlier that the envelope, $g(x)$, of the family of linear functions generated by $f(x)$ must satisfy Clairaut's equation, $g(\beta) - \beta g'(\beta) = f(g'(\beta))$. In [3] we proved the following theorem which gives the explicit form of the envelope $g(x)$ for a certain class of functions $f(x)$.

Theorem 2. *Suppose that $f(x)$ is a differentiable function such that $f'(x)$ is injective. $f(x) \to g(x)$ if and only if $g(x) = f((f')^{-1}(-x)) + x(f')^{-1}(-x)$.*

We will now use Theorem 2 to find the envelope of the family of linear functions whose x and y intercepts are one unit apart. This family of linear functions is the set of linear functions of the form $\frac{x}{a} + \frac{y}{b} = 1$ where $a^2 + b^2 = 1$. Rewriting this we see that the family is the set of linear functions of the form $y = \mp\frac{\sqrt{1-a^2}}{a}x \pm \sqrt{1-a^2}$. Figure 3 in the Maple-V worksheet shows this family of linear functions. Furthermore, this figure suggests that the envelope of this family of lines is a hypocycloid.

In order to apply Theorem 2 we must rewrite the family in the form $y = cx + f(x)$. Let $c = \mp\frac{\sqrt{1-a^2}}{a}$ and solving for a we see $a = \pm\frac{1}{\sqrt{c^2+1}}$. Substituting, we see that the family is the set of linear functions of the form $y = cx \pm \sqrt{\frac{c^2}{c^2+1}}$. Let $f(x) = \sqrt{\frac{x^2}{x^2+1}}$ then

$$f'(x) = \sqrt{\frac{x^2+1}{x^2}}\frac{x}{(x^2+1)^2}.$$

For $x > 0$ we see that $f'(x) = \frac{1}{(x^2+1)^{3/2}}$ and for $x < 0$ we see that $f'(x) = \frac{-1}{(x^2+1)^{3/2}}$ and that on each of these intervals $f'(x)$ is injective. Thus Theorem 2 applies. A straight forward calculation shows that $(f')^{-1}(x) = \sqrt{\frac{1-x^{2/3}}{x^{2/3}}}$ for $x > 0$ and $(f')^{-1}(x) = -\sqrt{\frac{1-x^{2/3}}{x^{2/3}}}$ for $x < 0$. Thus $(f')^{-1}(-x) = -\sqrt{\frac{1-x^{2/3}}{x^{2/3}}}$ for $x > 0$ and $(f')^{-1}(-x) = -\sqrt{\frac{1-x^{2/3}}{x^{2/3}}}$ for $x < 0$. In both cases it can be shown that $g(x) = f((f')^{-1}(-x)) + x(f')^{-1}(-x) = (1 - x^{2/3})^{3/2}$.

Thus we have shown that $\sqrt{\frac{x^2}{x^2+1}} \to (1-x^{2/3})^{3/2}$. Applying Theorem 2 we see that $-\sqrt{\frac{x^2}{x^2+1}} \to -(1-x^{2/3})^{3/2}$. Let $y = \pm(1-x^{2/3})^{3/2}$, then eliminating the radical we see that the envelope of the set of linear functions whose x and y intercepts are one unit apart is the hypocycloid $x^{2/3}+y^{2/3}=1$.

References

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