UNDERSTANDING TAYLOR'S THEOREM WITH THE *TI-85*

Gary H. Ford
Radford University
Box 6942, Radford University, Radford, Va. 24142
gford@.runet.edu

Background:

Three years ago, I joined the growing number of mathematics professors supporting reform in the teaching of calculus. I switched from the Anton text to the Ostebee-Zorn St. Olaf text being developed and have used these materials ever since. My choice of technological support was the *TI-85* graphing calculator. At the time I made the decision to go with the *85*, I actually would have preferred using MAPLE; however, Radford University, being a state supported school, had no money to purchase the computers and site license needed for that option. In hindsight, I believe that the *85* is actually the best choice I could have made, most notably because of its power and portability. I subsequently managed to secure some grant monies for our department to purchase 52 *TI-85*'s and worked out a deal with our bookstore to rent these units and some the bookstore had purchased to our students on a semester by semester basis. Several of our students use the rental units; but the majority recognize their importance and purchase their own units before the end of the fall semester. I have found the instrument extremely beneficial both in the classroom and for assignments and tests. The machine has changed the way I give tests. I now almost always include a take home portion of the test valued at somewhere from 15 to 30 percent of the test. My students are encouraged to use the graphing calculator on both parts of the test. And because they all have access to a graphing calculator, many of the examples and problems I use are much more "interesting" than I used to give.

In this paper, I will show some strategies for using the *TI-85* for teaching the idea of approximating functions with Taylor polynomials(including error bounds).

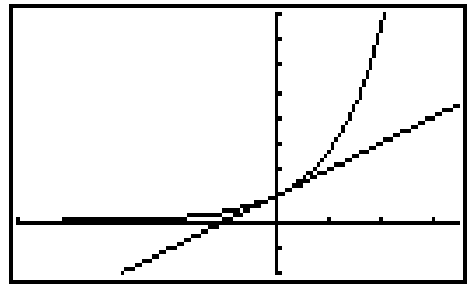
The Theorem:

The Taylor polynomial of degree n centered at x = 0 for function f is given by $p_n(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$, where $a_k = \frac{f^{(k)}(0)}{k!}$ and $f^{(k)}(0)$ is the k^{th} derivative of f evaluated at x = 0.

Motivation:

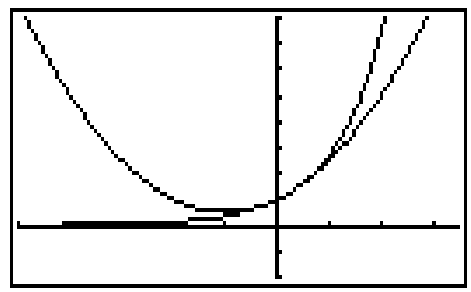
We would motivate the theorem with an example. One which works well is $f(x) = e^x$. Draw the graph of f and construct the tangent line at x = 0. My students already know this line as the *linear approximation* of the function f. We then would state(as we have many times before with this construction) that the function and tangent line are identical at x = 0 and close to each other *"near by"*. The graph, as it appears on the *TI-85*, is given below.

After discussing how a straight line cannot approximate a curve very well, I would lead them to the conclusion that a curve with not only the same tangent but also the same concavity near $x = 0$ would probably give a better approximation. We would then construct $q(x) = a_0 + a_1x + a_2x^2$, the *quadratic approximation* of e^x, where we want: $q(0) = f(0)$, $q'(0) = f'(0)$, and $q''(0) = f''(0)$. But e^x and all of its derivatives at $x = 0$ are 1. So we have that $q(0) = 1$, $q'(0) = 1$, and $q''(0) = 1$. Furthermore, since $q(x) = a_0 + a_1x + a_2x^2$, $q'(x) = a_1 + 2a_2x$ and $q''(x) = 2a_2$. Thus by substitution, $q(0) = a_0$, $q'(0) = a_1$ and $q''(0) = 2a_2$; this last equation says that $a_2 = ½q''(0)$. Thus: $a_0 = 1$, $a_1 = 1$, and $a_2 = ½$. Therefore the quadratic approximation is the polynomial: $q(x) = 1 + x + ½x^2$.



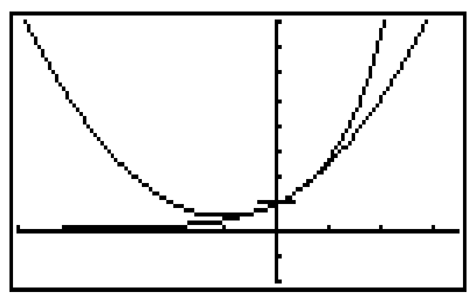
e^x and the linear approx

We would then sketch $f(x)$ and $q(x)$ on the same graph:



e^x and the quadratic approx

Since my students would have already dealt with both the linear and quadratic approximations of a function, these should present no problem. In addition, they have dealt with error bounds with such objects. We would then use the *TI-85* to find (albeit approximately) the interval on which the two functions are within, say, .01 of each other. One must look closely at the graph to see the horizontal line segment at height 1 near the vertical axis; this line segment is over the interval in question.



$|e^x-q(x)|\leq.01$

The keystrokes for this comparison: Under the assumption that f(x) is in equation y1 and q(x) is in equation y3, go to a *new* equation line and enter

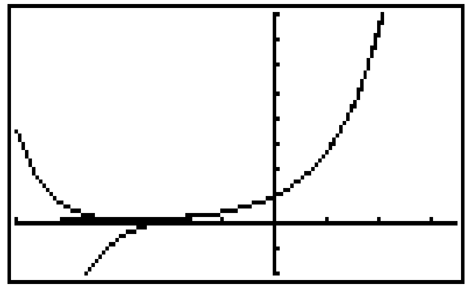
abs(y1-y3)≤.01

(you will find abs [the absolute value function] at the top of CATALOG and ≤ is under the TEST menu; its action is to graph at height 0 when the condition being tested is false and at height 1 when the condition is true. Note that the DrawDot format is better than DrawLine for this comparison since DrawDot will avoid a "vertical" connection between the x-axis and the line segment at height 1. You can then use TRACE on this new equation to approximate the endpoints of the interval where the relation is true.)

With the current range settings, TRACE reveals that the functions are within .01 of each other on the *(approximate)* interval (-.369,.345).

Taylor Polynomials:
At this point the students should be ready to generalize the idea that the more derivatives of the poly that are equal to the corresponding derivatives of f(x), the better the approximation should be. It is now time to do Taylor in earnest, both algebraically and graphically.

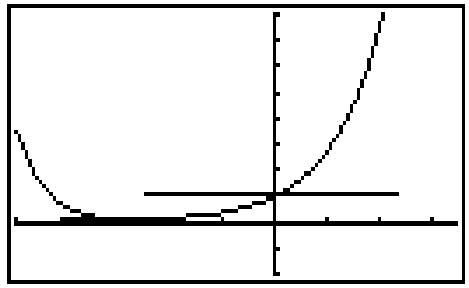
Examples:
We would continue the e^x example and build several Taylor polys, say through p_8. Here is a sketch of e^x with p_5 and p_8:



e^x with p_5 and p_8

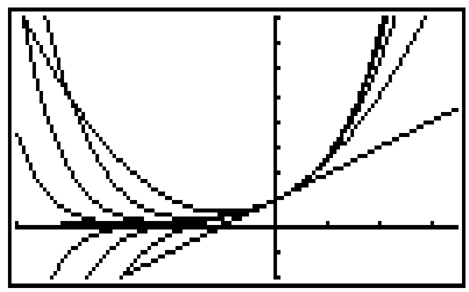
And don't forget to check how the interval on which the function and the Taylor poly are within .01 of each other has grown. Here is the comparison of e^x and $p_8(x)$:

These functions are within .01 of each other on (-2.504,2.353) now.



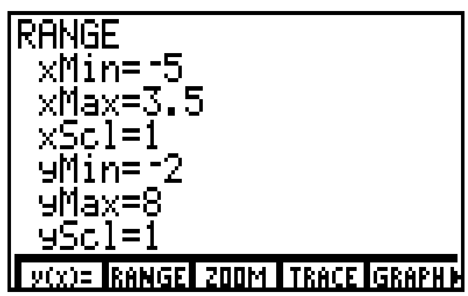
$|e^x-p_8(x)|\leq .01$

It is also very instructive to show f(x) with the various Taylor polys overlaid simultaneously. Here is e^x together with all the Taylor polynomials $p_1(x)$ through $p_8(x)$:



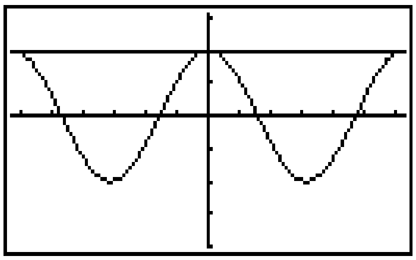
e^x with p_1-p_8

The window one uses for a demonstration of this type is obviously up to the individual instructor. I have included the range settings that I used for this example.

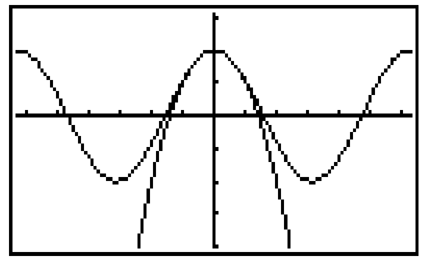


Viewing Window

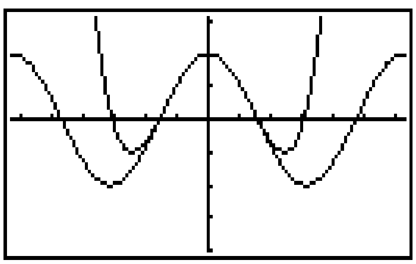
Assignment:
I would now ask my students to compute say the Taylor polynomials through say degree 12 for f(x) = 2cos x at x = 0 and to sketch the graphs. (The multiplier 2 is not necessary, but it gives a bit more definition to the curves without the student having to think about showing it.) [Actually, this is my favorite example; but since the odd powered Taylor polys vanish at x = 0, I don't think it is a good *first* example.] I would also have them check for the approximate interval on which f and the Taylor polys are within, say, .001 of each other, expecting results like those given below.



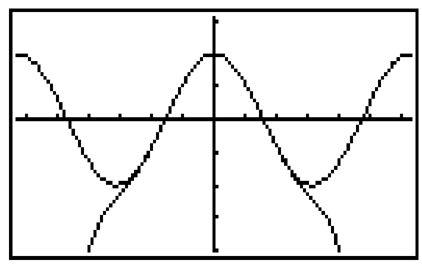
cos x with $p_1(x)$



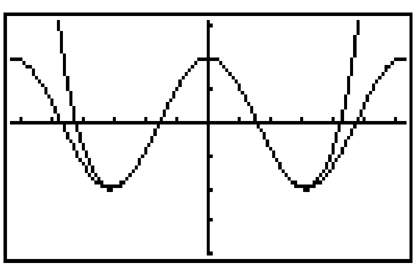
cos x with $p_2(x)$



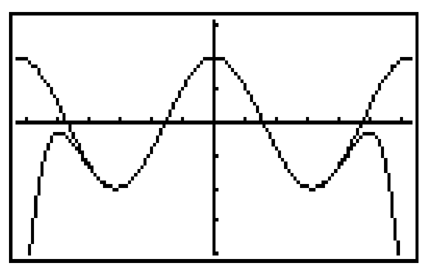
cos x with $p_4(x)$



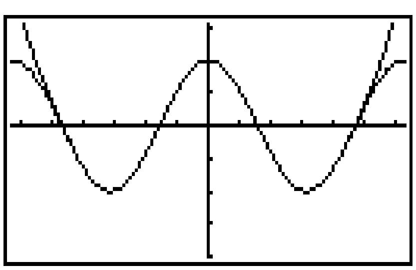
cos x with $p_6(x)$



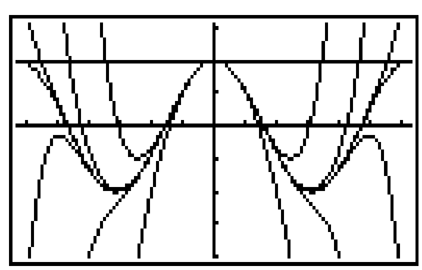
cos x with $p_8(x)$



cos x with $p_{10}(x)$

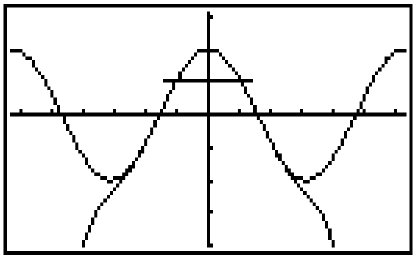


cos x with $p_{12}(x)$

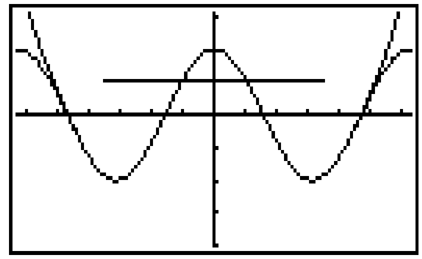


cos x with p_1-p_{12}

Testing the error to the .001 level on p_6 and with p_{12}:



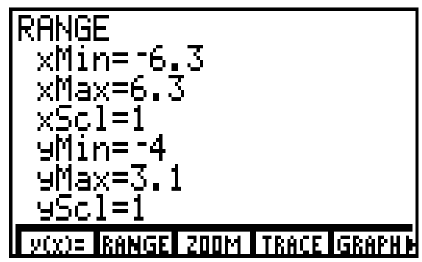
$|\cos x\text{-}p_6(x)| \le .001$



$|\cos x\text{-}p_{12}(x)| \le .001$

These graphs say that cos x and the 6th degree Taylor polynomial are within .001 of each other on the (approximate) interval (-1.4,1.4), while cos x and the Taylor 12th degree polynomial are within .001 of each other on the (approximate) interval (-3.5,3.5).

The window for all the graphs above:



Viewing Window

Direction:
The extension of this idea, finding Taylor polynomials for functions centered at a general x = a, would be natural graphically since my students have done much work with horizontal translations. And with this foundation, we are then well on our way to *understanding* what it means to say that a function f(x) can be expressed as an infinite series--something which took me quite a while to come to grips with.

References

Alan Levine and George Rosenstein, Discovering Calculus, Volume II, Preliminary Version, McGraw-Hill, Inc., 1994.

Arnold Ostebee and Paul Zorn, Calculus From Graphical, Numerical, and Symbolic Points of View, Prelim Ed., Saunders College Publishing, 1993.