ITERATION IN FIRST SEMESTER CALCULUS

James A. Walsh
Oberlin College
Department of Mathematics, Oberlin, OH 44074
jimw@cs.oberlin.edu

Introduction. Dynamical systems is the study of any process that changes with time. Within this broad classification are the two branches of dynamics: the continuous and the discrete. Continuous dynamical systems involve the study of physical systems with a continuous time variable, as in the case of a simple pendulum moving under the influence of gravity. Such systems are often modelled by differential equations, and one is interested in understanding (at least qualitatively) solutions to these equations. Discrete dynamical systems are concerned with models of processes which vary at regular, fixed time intervals. For example, fluctuations in the stock market on a daily basis or the spread of measles in a given city on a monthly basis represent discrete dynamical systems. Continuous dynamical systems can often be reduced to discrete systems by sampling solutions to the differential equations at multiples of a fixed time interval (and in more general ways as well; see [5, §1.5]). Discrete dynamical systems, or *iteration* of an appropriate model function, thus arise naturally in our attempts to understand the real world.

This paper presents a brief summary of a week long module on iteration that I incorporate into the first semester calculus curriculum. The connection between iteration and first semester calculus is very natural. As the reader will see, the derivative plays the leading role in understanding the long term behavior of iterates of a function. Secondly, Newton's method is a classical iterative scheme which exhibits surprisingly complicated dynamics when viewed on a global scale, i.e., away from the zeros of the function in question. For these two reasons (and because it's fun!), iteration fits very naturally into the first semester calculus curriculum.

Iteration. Let $F : \mathbf{R} \to \mathbf{R}$ be a function defined on the real line. Given an initial point x_0, the *orbit* of x_0 is the sequence

$$x_0,\ F(x_0),\ F(F(x_0)) = F^2(x_0),\ F(F(F(x_0))) = F^3(x_0), \ldots, F^n(x_0), \ldots.$$

The simple question one would like to answer is "What happens to an orbit over time, or as $n \to \infty$?" At times this question is as easily answered as it is posed. If $F(x_0) = x_0$, the orbit of x_0 is $x_0,\ x_0,\ x_0, \ldots$. Such a point x_0 is called a *fixed point* of F. If for some point x_0 and a smallest positive integer n we have $F^n(x_0) = x_0$, the orbit of x_0 simply repeats or cycles every n iterates. Such a number x_0 is called a *periodic point* of period n for F. What other types of behavior are possible? As the students realize very quickly, a process as simple as iteration can lead to amazingly complicated long term behavior. The students are required to investigate these possibilities through the following three computer lab assignments, which comprise the week's homework assignments.

Lab 1. We begin with the two parameter family of linear functions $F(x) = ax+b$, where a and b are real parameters. The students are introduced to an electronic spreadsheet in class, then asked to determine the fate of all orbits for this linear family using the spreadsheet in the computer lab. This is not as daunting a task as one might at first suspect; experimentally most students arrive at the conclusion that if $|a| < 1$ all orbits converge. If $|a| > 1$ then all orbits, with the exception of the unique fixed point, diverge to infinity. The reader is left with the case $|a| = 1$. The simplicity of the long term behavior of the orbits is ideal for this first lab as the students are also learning how to use the spreadsheet.

One astute student showed by induction that, for this linear family,

$$F^n(x_0) = a^n x_0 + b\sum_{i=0}^{n-1} a^i = a^n x_0 + b\Big(\frac{1-a^n}{1-a}\Big).$$

Hence, if $|a| < 1$, $F^n(x_0) \to b/(1-a)$ as $n \to \infty$, where $b/(1-a)$ is the fixed point of F. Likewise, if $|a| > 1$ the orbit of x_0 diverges to ∞. We conclude Lab 1 with a discussion relating iteration of linear functions with models of real life discrete dynamical systems. In particular, simple linear growth/decay population models, and a model for the principal in a bank account given a fixed interest rate compounded monthly, subject to monthly deposits/withdrawals, are discussed as members of the linear family $F(x) = ax + b$. Thus iteration determines the long term behavior for these models.

Lab 2. It is natural to next consider families of quadratic maps. Motivated by a population model for a species in an environment with limited resources [2, §1.1], we consider the one parameter *logistic* family $F_a(x) = ax(1-x)$. The real parameter $a \in [0, 4]$ is called the growth factor, while $x \in [0, 1]$ represents a percentage of a limiting population level for the given environment. The question once again is "What happens to orbits over time?", and in this case the answer is not so apparent!

In Lab 2 students are provided with a list of a values and asked to determine via the spreadsheet the long term behavior of most orbits. For $0 \le a < 1$ they find that orbits tend to 0, so that the population becomes extinct over time. For $1 < a < 3$ however, orbits converge to a unique, non-zero value which increases with a. Now letting a run from 3 to 4 students encounter the "period doubling route to chaos" [2, §1.17], including a values for which the orbit of x_0 is apparently a random list of numbers. The dynamics for the logistic family are strikingly complicated and students are not expected to classify all possible long term behaviors. They are required, however, to make a (rough) *bifurcation plot*, in which the limiting behavior of orbits is plotted versus the parameter a. A computer generated bifurcation plot is presented in Figure 1. We then use commercially available software to investigate the wonderful richness of the bifurcation plot, including a brief discussion of Feigenbaum's universal constant [3, §10.4].

What is the connection between the above investigations and first semester calculus? Fix $a \in (1, 3)$, and let $x = p$ be the limiting value of all orbits for that a value. Note that p must be a fixed point for F_a as it is the limit of the sequence of iterates of the continuous function F_a. Since orbits converge to p as $n \to \infty$, p is called an *attracting* fixed point.

What is the source of this "attraction"? Students have seen that the tangent line to the graph of a function at a given point is the best local linear approximation for the graph near that point. Thus, given an x_0 near the fixed point p, the orbit of x_0 under F_a should behave roughly like the orbit of x_0 under the linear map corresponding to the tangent line at the point (p, p). This linear map is given by the equation $L(x) = F_a'(p)(x-p)+p$,

Figure 1. The bifurcation plot for the logistic family $F_a(x) = ax(1-x)$.

which has $x = p$ as its unique fixed point. From Lab 1 we then have that if x_0 is close enough to p its orbit under F_a converges to p if $|F_a'(p)| < 1$. One readily computes that $p = (a-1)/a$ and $|F_a'(p)| = |2-a| < 1$ since $a \in (1, 3)$. Thus it is the derivative of F_a at the fixed point $x = p$ (and of F_a^n at the periodic point p of period n for the attracting periodic orbits visible in Figure 1) which governs the local behavior of orbits near the fixed (or periodic) point. The students are led to the following theorem.

Theorem. Suppose $F : \mathbf{R} \to \mathbf{R}$ *has a continuous derivative. If* $F(p) = p$ *and* $|F'(p)| < 1$ *there is an open interval* I *containing* p *such that for any* $x \in I$, $F^n(x) \to p$ *as* $n \to \infty$.

Proof. Since $F'(x)$ is continuous, pick $\epsilon > 0$ and $A > 0$ so that $|F'(x)| \leq A < 1$ for $x \in (p-\epsilon, p+\epsilon)$. Suppose $|x-p| < \epsilon$. By the Mean Value Theorem there exists c between x and p so that $|F(x) - F(p)| = |F'(c)||x-p|$. Since $F(p) = p$, we have

$$|F(x) - p| \leq A\,|x-p| < |x-p|.$$

Thus, $F(x) \in (p-\epsilon, p+\epsilon)$, and one can repeat the argument. By induction we have $|F^n(x) - p| \leq A^n|x-p|$, implying $F^n(x) \to p$ as $n \to \infty$. ∎

Remark. If $F(p) = p$ and $|F'(p)| > 1$ there is an interval I containing p such that the orbit of any $x \in I$, $x \neq p$, eventually leaves I. Such a point p is called a *repelling* fixed point. Thus, for a fixed a slightly larger than 3, F_a has an attracting cycle of period 2 as in Figure 1. The fixed point $p = (a-1)/a$ still exists, but $|F_a'(p)| > 1$, and hence it is no longer attracting and does not appear in the bifurcation plot.

Lab 3. After introducing Newton's method in class as an iterative process used to approximate solutions to equations $F(x) = 0$ students are given Lab 3. For this assignment students must investigate the dynamics of Newton's method for the one parameter family of cubics $F_c(x) = (x+2)(x^2+c)$, $c \in \mathbf{R}$, with associated Newton's method map $N_c(x) = x - (F_c(x)/F_c'(x))$ (see also [1]). This is more of an open ended assignment as first semester calculus students cannot be expected to completely work this out on their own (perhaps this lab would be better assigned as a group project). They can however make progress in understanding the behavior of orbits under N_c in the case $c < 0$, when F_c has three real roots. When taken from a global perspective (what happens to *all* orbits on the real line?) the dynamics, while not as complicated as in Lab 2 with the logistic family, are still quite interesting [6]. The students are also provided with leading questions with which to investigate the $c > 0$ case, and surprisingly they again find attracting fixed points, two cycles, four cycles, etc. as c is varied. This naturally leads to the consideration of a bifurcation plot for the one parameter family N_c, which in this case is a one parameter family of *rational* maps. It turns out that if N_c has an attracting periodic cycle the orbit of at least one critical point must converge to this attracting cycle [4]. A computation yields

$$N_c'(x) = \frac{F_c(x)F_c''(x)}{(F_c'(x))^2},$$

so the critical points of N_c are the zeros of F_c and $x = -2/3$. Note that $F_c(p) = 0$ implies $N_c(p) = p$, so that zeros of F_c are fixed points for N_c. Thus to draw the bifurcation plot in Figure 2 one plots the parameter c versus the limiting behavior of the orbit of $x = -2/3$. Surprisingly we see the quadratic bifurcation plot buried within Figure 2; there are infinitely many copies, several of which can be discerned in Figure 2. This suggests that the period doubling route to chaos is a rather ubiquitous phenomenon in systems undergoing transitions to chaos.

What makes Newton's method work if one begins sufficiently close to a point p satisfying $F(p) = 0$? Students show that for such a p, $N(p) = p$ and $N'(p) = 0$, so that zeros of F are attracting fixed points for the Newton map N. We conclude with a discussion of how the rate of convergence of Newton's method to p is influenced by $N'(p) = 0$. That is, we discuss why Newton's method converges so quickly when it in fact does converge to a zero of F.

Conclusion. In this module first semester calculus students are excited to see how simple processes often have very complicated consequences. Iterating a function is a simple process, yet the results are often very complex. Iteration is also well suited for first semester calculus as students come to realize that the derivative at a fixed or periodic point determines to a large extent the behavior of orbits over time. The view of the derivative as locally contracting or expanding intervals is a nice alternative to its traditional interpretation as the slope of the tangent line to the curve. Students also see Newton's method in a new light and are surprised and fascinated by the intricacies of the dynamics. From student responses this week long module on discrete dynamical systems is a success and is one that I will continue to present in future sections of first semester calculus.

Figure 2. The bifurcation plot for the family of Newton maps N_c. As evidenced by the zoomed in plot on the right, iterates of N_c are "quadratic like" for certain ranges of c values.

Acknowledgments. Computer Labs 1 and 2 are slight modifications of labs developed by Robert L. Devaney, with whom I worked on a project aimed at teaching the ideas of iteration to high school teachers. I thank him for allowing me to use these labs. I also thank Paul Blanchard for suggesting the family F_c in Lab 3.

References.

[1] Berkey, D. and Blanchard, P., *Calculus*, 3rd ed., Saunders, Philadelphia, 1992, Appendix 4.

[2] Devaney, R.L., *An Introduction to Chaotic Dynamical Systems*, 2nd ed., Addison-Wesley, Reading, MA, 1989.

[3] Devaney, R.L., *A First Course in Chaotic Dynamical Systems*, Addison-Wesley, Reading, MA, 1992.

[4] Fatou, P., Sur les équations fonctionelles, *Bulletin Société Mathématique de France* 45 (1919), 161-271.

[5] Guckenheimer, J. and Holmes, P.J., *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.

[6] Walsh, J.A., The dynamics of Newton's method for cubic polynomials, *The College Mathematics Journal* 26 (1) 1995.