FINDING DERIVATIVES WITHOUT THE NOTION OF LIMITS

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Calculus reform has been an issue of much national concern and attention for many years. In 1986 a conference/workshop and in 1987 a national colloquium led to important publications such as *Toward a Lean and Lively Calculus* [1] *and Calculus for a New Century* [2]. This led to the Calculus Reform Initiative, sponsored by the National Science Foundation, and there has evolved a national movement to reevaluate the goals, content and pedagogy for the teaching of calculus. Results from this initiative can be found described in *Priming the Calculus Pump,* [3] and in works of Linn [4] and Leinbach[5]. Recently, Nemirovsky [6] has stated: "Most calculus courses do not go beyond me students' acquisition of procedures and notations which are quickly forgotten. As a consequence, the understanding of calculus that students. require for science and engineering majors are simply not sufficient channels to address seriously the need to incorporate the insights of calculus into education more generally. Moreover, these courses are directed toward only that small subset of the population." Recall that the 1989 National Research Council report, Everybody Counts,[7] stated, in part: "Calculus is a powerful and elegant example of a mathematical method, leading both to major applications and major theories. The language of calculus has spread to all scientific fields; the insight it conveys about the nature of change is something that no educated person can afford to be without."

The purpose of this paper is to introduce an approach to calculus which avoids the concept of a limit until rational function calculus is developed. It is the belief and experience of the authors that the major stumbling block, which prevents a wide population of students from a successful encounter with calculus, is that single notion of 'limit' which is so foreign to most students' intuition. The approach presented herein is rigorous and depends on concepts and definitions introduced by the old master teacher, R.L. Moore[8] in the 1930's and more recently, application of those concepts and definitions to polynomial functions by Roman[9]. This approach to teaching calculus has been successfully used midi leaches of graded 6-12 in Eisenhower mathematics summer projects and in three semesters of *(Calculus for Business Majors* at Incarnate Word College.

Though we will use the notation of functions, we begin with Moore's definition of a simple graph: The statement that the graph *G* is simple means that *G* is a set of points in the plane such that no vertical line contains two points of *G*. It follows immediately that if *G* is a graph of a function, then *G* is a simple graph. Indeed, by introducing simple graphs first the notion of a function being a collection of ordered number pair, no two pair having the same first term, is more easily understood by the student. Moore's definition of slope of a simple graph depends on the notion of slope of a straight line, but students at a very early age master and understand that each non-vertical line has associated with it one and only one number which is called the slope of that line. In Roman's paper [10], it is seen that the uniqueness of slope can be established using only algebra. By talking about the slope of a simple graph, using only concepts no more complicated than the slope of a straight line and the concept of that two intersecting lines form an angle at the point of intersection, the student can find derivatives of functions using nothing more sophisticated than algebra.

Moore's definition of slope of a simple graph is fundamental to the rigor of this approach to the teaching of calculus. However, if the intent of the course is to teach calculus without engaging in the proof of theorems (as in calculus for business majors), this definition may not be stated at all and the teacher would simply discuss the slope of the line which is tangent to the graph. In other courses, such as a course which is a mathematics requirement of liberal arts majors, the definition should be stated and an investigation of properties of graphs with slope should be made. Moore's definition of slope of a simple graph G is in terms of a single point of G: The statement that the number m is the slope of the simple graph G at the point A of G means that

(1) if h and k are vertical lines, with A between them, then some other point of G is between h and k, and

(2) if L is a line which contains A and L has slope m, then it is true that if α is any acute angle with vertex at A and a point of L in the interior of α , then there must exist vertical lines, h and k , with A between them, such that every point of G between h and k , except A, is in the interior of α, or the angle vertical to α .

The first condition prevents A from being an isolated point of G. The second condition is at the heart of the definition of slope (or equivalently, derivatve). We find that it is best to allow students to experiment with graphs of functions with which they already are acquainted and with which they have achieved a comfort level. Consider the grapf of the function $f(x) = x^2$.

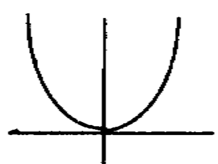

Fig. 1

$f(x) = x^2$

The first question to raise to the student is to determine whether the graph has slope at the origin and, if so, what is it? The x-axis often is the first guess and the students already know that horizontal lines have slope zero. So the conjecture is that the graph has slope 0 at the origin. The next question to answer is whether every non-horizontal and non-vertical line, which contains the origin, must intersect the graph in more than one point. Here, students should be invited to use the graphing calculator to investigate the possibilities. Normally someone will notice that any line, having slope and containing the origin, must be described by $f(x) = mx$ for some number m. Since the equation under consideration deals only with nonhorizontal lines, then m is not 0. So now the question is reduced to solving the equation $x^2 = mx$, and it is easy to recognize that m satisfies that equation. So, every non-horizontal, non-vertical line containing the origin must contain two points of the graph of $f(x) = x^2$.

Consider the line L_1:

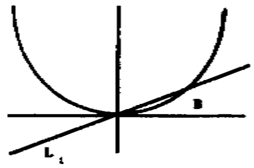

Fig. 2 - First Side of Angle

Call the point of intersection of L_1 and the graph: B.

Consider the line L_2:

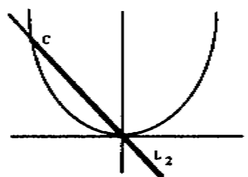

Fig. 3
Second Side of Angle

Call the point of intersection of L_2 and the graph: C. It is clear that L_1 and L_2 form an angle with vertex at the origin and it is equally clear that some point of the x-axis is in the interior of that angle:

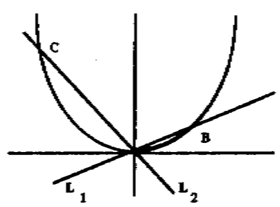

Fig. 4
Angle Formed by L_1 and L_2

Consider the vertical lines which contain the points B and C, respectively:

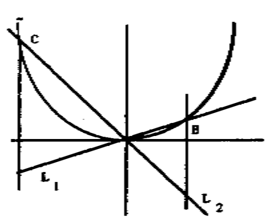

Fig. 5
Acute Angle with Vertical Lines

These two vertical lines clearly satisfy the second condition of Moore's definition of slope of a simple graph, so it follows that the slope of this graph at the origin is zero.

Let us continue our investigation of the graph of $f(x)=x^2$. We pick a point of the graph, say A and assume there is a line, L, containing A such that L satisfies Moore's definition of slope at the point A. (Refer to fig. 6 for illustration). Then L must be described by an equation such as $y=mx + b$ for some number m and some number b. Then, in functional notation, A is described as both [x, $f(x)$] and [x, y] and it follows that y= $f(x)$ and, at **A,** $mx + b = x^2$. From this last equation it follows that $0 = x^2 -mx - b$.

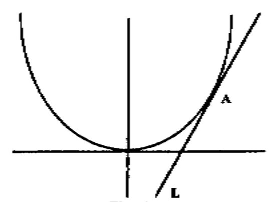

Fig. 6
Slope line **L** to the graph of $f(x) = x^2$ at A

Since L and the graph of $f(x) = x^2$ have only one point, A, in common, it is immediate that $0=x^2 - mx - b$ has only one root and, thus, must be a perfect square, or of the form $(x + c)^2$ for some number c. But from algebra the student knows that c $= - m / 2$ [and also that $b = (-m/2]$. Using only algebra, it follows that $0 = x^2 - mx - b =[\ x - m/2]^2$ and this is possible only if $x = m/2$. But then, $m = 2x$ and we have found the slope of the line which contains A and, indeed, the slope of the graph of $f(x) = x^2$ at **A** is $2x$.

At the outset of this investigation of the graph of $f(x) = x^2$, we assumed that the graph had slope, so we have established that, if the graph has slope, then that slope is $2x$. This presents an opportunity for the student to become better acquainted with the graphing calculator. Let the students pick any point on the graph. For our purposes we will pick (2,4). From what we have done, we expect the slope of de graph at dmi point to be 2x = 2 **x** 2 = 4. Now ask the students to find the equation of the line which contains the point (2,4) and has slope 4. That equation is

$y = 4x - 4$. Now let the students get a "feel" for this geometric definition of slope, testing the definition at the point (2,4).

Refer to the illustration below:

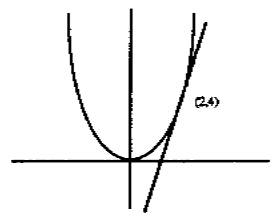

Fig. 7
Slope Line

Now let the students construct two intersecting lines, each containing the point (2,4) so that a point of the slope line is contained in the interior of the angle formed by the intersecting lines:

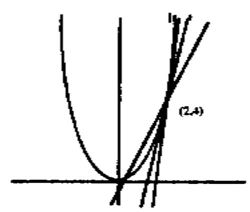

Fig. 8
Two lines intersecting at (2,4) with slope line

They should be allowed to select two slope numbers, one greater than 4 and the other less than 4, to find the equations which describe two lines which contain (2,4), one line having slope greater than 4 and the other having slope less than 4, as shown in Fig. 8. The student should determine where vertical lines should be drawn to satisfy the geometric definition of slope. It is likely that the student will have to use the "zoom" feature to be able to see clearly enough where to take the two required vertical lines. Once the position of the two vertical lines is determined, there is graphical reassurance that $y = 4x - 4$ is the correct slope line. As other pairs of lines are selected, with respective slopes closer to 4, the student can use the graphing calculator to gain additional reassurance. Indeed, this exercise prepares the student to grasp more easily the later introduction of the concept of "limit".

Now we continue with algebraic definitions of derivatives by considering the graph of the function $f(x) = 1/x$, $x \neq 0$. Consider a candidate slope line to the simple graph of that function, as illustrated below:

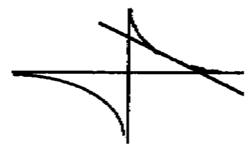

Fig. 9
Slope Line to $f(x) = 1 / x$

The slope line must have a descriptive equation such as y = $mx + b$, and since there is a point common to the slope line and the graph, $1 / x = mx + b$. From algebra, it is seen that since $1 / x = mx + b$, then $1 = mx^2 + bx$ and, in turn, that becomes $0 = mx^2 + bx - 1$. Then $0 = x^2 + (b / m)\, x - 1 / m$. Again, since there is only one point in common to the slope line and the graph, the expression $x^2 + (b / m) - 1 / m$ must be a perfect square and of the form $(x + c)^2$ for some number c . But, from algebra, the student knows that $c = b / 2m$ (and also that $[(b / 2m)^2 = (-1 / m)]$. We will use algebra to attempt to solve for m in the equation $0 = x^2 + (b / m)\, x - 1 / \mathrm{m}$. From above, since $(b /2m)^2 = 1 / m$ (m must be negative) then $b^2 /(4m^2) = -1 / m$, and $b^2 /4m = - 1$. Then $b^2 = -4m$ and $b = \sqrt{-4m} = 2\sqrt{-m}$ (remember that m is negative). Now we can substitute for b in the equation $0 = x^2 + (b / m)x - 1/ m$ to get: $0 = x^2 +2\, [(\sqrt{-m}) /m] - 1 / m = x^2 + [(2 / \sqrt{-m}]\ \ x - 1 / m = (x + 1 / -m)^2$ so $x = 1 /\sqrt{-m}$ and it is seen that $m = -1 / x^2$.

Finally, consider the function $f(x) = \sqrt{x}$:

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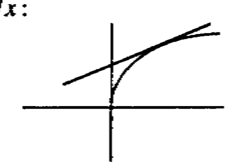

Fig. 10
Slope of Line to graph of $f(x) = \sqrt{x}$

From the illustration above (which the student can gain from the graphing calculator), it can be seen that if there is a slope line to the simple graph of the function, then the slope line must be of the form y = $mx + b$ and, since the slope line and the graph have a point in common, then $mx + b = \sqrt{x}$. Then $mx - \sqrt{x} + b = 0$ and it follows that $0 = x + \sqrt{x} /m + b / m$. This also must be a perfect square, with only one root, and it is possible to solve for m. Using only algebraic techniques of high school algebra, it can be seen that $m = 1/2\sqrt{x}$.

As in the first example given herein, the student can utilize the graphing calculator to become more adept with that new instructional device, but even more importantly, the student uses only algebra to find the slopes of simple graphs. We need not tell the student that the slope of a simple graph is really a derivative of a function, until the student has considerable comfort with the process. In this paper, we provide only an introduction to this approach to the teaching of calculus without the usual calling on limits. The examples chosen suggest how the slope of a graph of a polynomial function can be found, using only algebra. Indeed, this approach can be extended to all rational functions; that is, a function which is the quotient of two polynomial functions, the denominator function not zero.

It should be noted that, with this approach, and with only the least upper bound principle, the student can have success in solving maximum-minimum problems ... without ever using the word "derivative" or dealing with the concept of "limit".

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