# An Algebraic Approach to the Immutability of the Slope of a Line

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For the past forty years educators have been grappling with the idea of changing the calculus curriculum to what is called a "leaner and livelier" calculus[1]. Ever since the Committee on the Undergraduate Program in Mathematics (CUPM) issued a call for revamping in 1953[2], mathematicians have responded by "getting grants, setting up laboratories, developing software, assigning student projects, writing their own textbooks, and rethinking from top to bottom what should go into a calculus course"[3]. All of this energy and activity directed toward the calculus curriculum has admittedly changed the look of the calculus classroom from paper and pencil tedium to what some call high tech fanfare. However, the syllabus of the calculus class evidences little change. Most still begin with the concept of a limit and build upon it in a somewhat systematic and traditional manner[4]. The purpose of this paper is to introduce a unique method of dealing with the derivative of a function which circumvents the concept of limit and requires only a knowledge of high school algebra. The elimination of the concept of limit until a later point in the study of mathematics and approaching the derivative with a purely algebraic method can simplify one's entry into higher mathematics and enrich day-to-day living with a broader knowledge of the mathematics which pervade out technical world today.

## The Immutability of the Slope of a Line

In order to build a bridge from algebra to calculus, a student must turn first to the link between algebra and geometry: the Cartesian coordinate system. This system takes the algebra which was first worked with letters and numbers and gives a "picture" of the equation that has been manipulated. Beginning with the equation of a line, one is able to recognize that some lines travel "uphill" and some travel "downhill" giving each a certain "personality" on the grid system of the Cartesian coordinates.

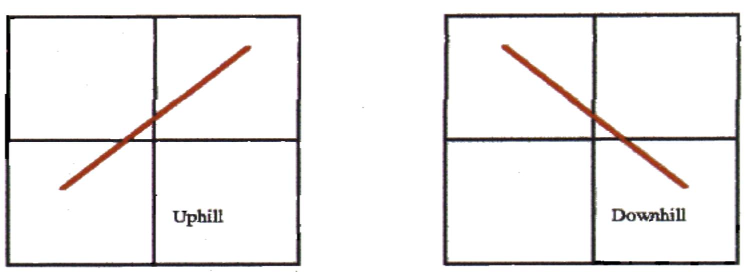



Fig. 1 *Uniqueness of Slope*

The algebra student who has been accustomed to dealing with an equation such as, $y = 2x - 1$, can easily see that substituting different values for *x* produces unique values for *y* and results in a pair of "coordinates" which lie on the line pictured as $y = 2x - 1$. For the value $x = 2$:

$$\mathbf{y = 2(2) - 1}$$
$$\mathbf{y = 4 - 1}$$
$$\mathbf{y = 3}$$

This generates the pair of coordinates (2,3) on the line $y = 2x - 1$.

For the value x = 7:

$$\begin{aligned} y &= 2(7) - 1 \\ y &= 14 - 1 \\ y &= 13 \end{aligned}$$

This produces the coordinates (7,13) on the same line.

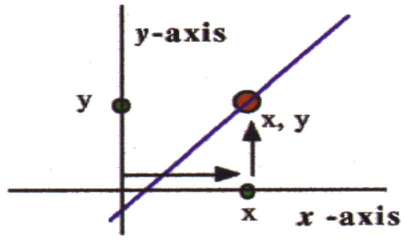



Fig.2 *Plotting ordered pairs*

Using these coordinates, a student is able to plot the points on the *x* and *y* axes and measure the distance between any two points on this line. He or she can also deal with a function. And he/she will be able to identify the "personality" or the slope of the line using a definition of the slope of a line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

An interesting aspect of the slope of a line is that no matter which two points or coordinates of real numbers are chosen on that line, the slope remains the same. A traditional proof of this characteristic of line involves similar triangles and relies on a fair bit of knowledge of geometry. However, the immutability or unchanging character of the slope of a line can be proved with only the knowlege of algebra. Before leaving an algebra course, most students have become familiar with the general equation for a line which involves coefficients for the *x* and *y* terms plus a constant [5]. This general form for the equation of a line is:

$$ax + by + c = 0$$

Taking four different values for the coordinates *(x,y)*, the following algebraic proof emerges.

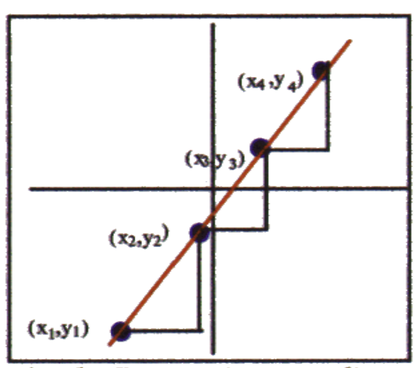



Fig. 3 *Four points on a line*

Substituting $(x_1,y_1)$ into the general equation and solving for $y_1$, results in:

$$\boldsymbol{ax_1 + by_1 + c = 0}$$
$$\boldsymbol{by_1 + c = -ax_1}$$
$$\boldsymbol{by_1 = -ax_1 - c}$$

$$y_1 = \frac{-ax_1 - c}{b}$$

Solving for $y$ using the other values for $x$ and $y$,

$$y_2 = \frac{-ax_2 - c}{b}$$
$$y_3 = \frac{-ax_3 - c}{b}$$

$$y_4 = \frac{-ax_4 - c}{b}$$

Remembering the definition of the slope of a line:

> If points $P_1$ and $P_2$ with coordinates $(x_1,y_1)$ and $(x_2,y_2)$ respectively, are any two different points on a line, the the slope of the line (denoted by M) is
>
> $$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and using the values for $y$ calculated above, one may subtract one value of $y$ from the other:

$$y_2 - y_1 = \frac{-ax_2 - c}{b} - \frac{-ax_1 - c}{b}$$

Finding a common denominator of $b$ and combining the numerator, the equation results in:

$$y_2 - y_1 = \frac{-ax_2 - c + ax_1 + c}{b}$$

Combining like terms gives:

$$y_2 - y_1 = \frac{-ax_2 + ax_1}{b}$$

Factoring out a $-a$ leaves:

$$y_2 - y_1 = \frac{-a}{b}(x_2 - x_1)$$

Dividing through by $(x_2-x_1)$ produces:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-a}{b}$$

This equation brings us back to the definition of slope and tells us that the value for the slope $-\frac{a}{b}$ is using the next values for $x$ and $y$, the results are as follows:

$$y_3 - y_2 = \frac{-ax_3 - c + ax_2 + c}{b}$$

$$y_3 - y_2 = \frac{-ax_3 + ax_2}{b}$$

$$y_3 - y_2 = \frac{-a}{b}(x_3 - x_2)$$

$$\frac{y_3 - y_2}{x_3 - x_2} = \frac{-a}{b}$$

The last two values for *x* and *y*, produce:

$$y_4 - y_3 = \frac{-ax_4 - c + ax_3 + c}{b}$$

$$y_4 - y_3 = \frac{-ax_4 + ax_3}{b}$$

$$y_4 - y_3 = \frac{-a}{b}(x_4 - x_3)$$

$$\frac{y_4 - y_3}{x_4 - x_3} = \frac{-a}{b}$$

Therefore, using just algebra one can prove that the slope of a line remains the same no matter which two points on the line are chosen.

$$M = \frac{y_2 - y_1}{x_2 - x_1} = -\frac{a}{b} \qquad \frac{y_3 - y_2}{x_3 - x_2} = -\frac{a}{b} \qquad \frac{y_4 - y_3}{x_4 - x_3} = -\frac{a}{b}$$

Fig. 4 Slopes

Carrying the problem a bit farther, we are able to show that the ratio of the distance from one point to the other is in the same ratio as the ratio between the sides of the triangles in the original "picture" of the line.
Taking the ratios above and cross multiplying, we can see that the *x* sides of the triangle are proportionate and the *y* sides are proportionate.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

$$(y_3 - y_2)(x_2 - x_1) = (y_2 - y_1)(x_3 - x_2)$$

$$\frac{y_3 - y_2}{x_2 - x_1} = \frac{x_3 - x_2}{x_2 - x_1}$$

$$\frac{y_3 - y_2}{x_3 - x_2} = \frac{y_4 - y_3}{x_4 - x_3}$$

$$(y_4 - y_3)(x_3 - x_2) = (y_3 - y_2)(x_4 - x_3)$$

$$\frac{y_4 - y_3}{x_3 - x_2} = \frac{y_4 - y_3}{x_3 - x_2}$$

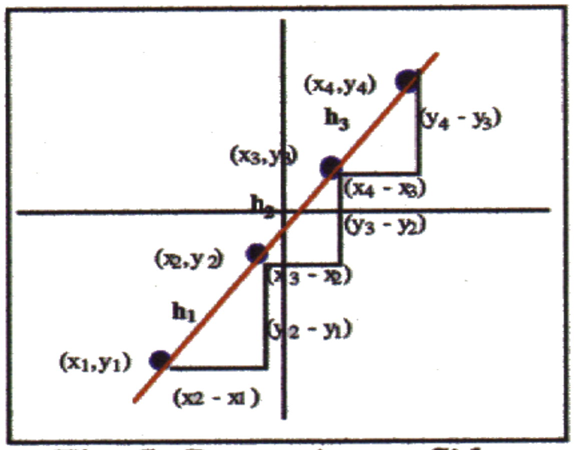
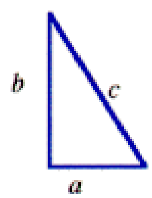


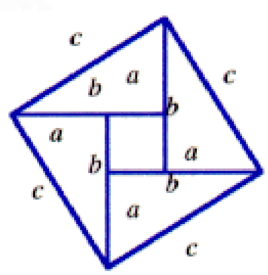

**Fig. 5** ***Proportionate Sides***

The Pythagorean Theorem would be useful at this point, because the distance between each point is actually the hypotenuse of the right triangles formed by the line and the changes in x and the changes in y at each point. Since the *x* and *y* sides to the triangles are proportionate, the Pythagorean Theorem can be used to show that the hypotenuse of each triangle is proportionate to the others. Pythagoras is given credit for the proof showing with *a* being the shortest leg of a right triangle, *b* being the longer leg, and *c* being the hypotenuse. Frank J. Swetz and T. I. Kao in their writing, *Was Pythagoras Chinese? An Examination of the Right Triangle Theory in Ancient China*[7] provide a simple proof of Pythagoras' theorem taken from an early Chinese text which dates back to 1100 B.C., 500 years before Pythagoras.

Begin with the right triangle.

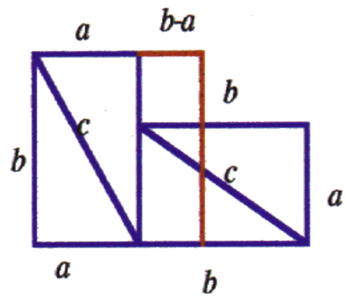



Arrange four such triangles as follows.




The area of the outer square is $c^2$. Since the long side of the triangle is *b*, each side of the inner square is *b-a*.

Now rearrange the four triangles and the inner square as




Now the figure makes up two squares: one with the sides of the length *a* and the other with sides of length *b*. The sum of the two squares in this figure is $a^2 + b^2$. Since the area of this figure is the same as the area of the preceding figure, then $\mathbf{c^2 = a^2 + b^2}$.

Knowing that $\mathbf{c^2 = a^2 + b^2}$, one may use the following algebraic approach to proving that all the sides of the triangles formed by the line in our picture and the changes of *x* and *y* are, in fact, in the same proportions.

Using the Pythagorean Theorem, one can see that the values for $h_1$ and $h_3$ are:

$$h_1 = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

$$h_3 = \sqrt{(y_4 - y_3)^2 + (x_4 - x_3)^2}$$

Dividing one hypotenuse by the other results in:

$$\frac{h_1}{h_3} = \frac{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}{\sqrt{(y_4 - y_3)^2 + (x_4 - x_3)^2}}$$

Then the question becomes:

$$\frac{h_1}{h_3} = \frac{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}{\sqrt{(y_4 - y_3)^2 + (x_4 - x_3)^2}} = \frac{x_2 - x_1}{x_4 - x_3} \ ?$$

Squaring both sides of the equation:

$$\frac{h_1}{h_3} = \frac{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}{\sqrt{(y_4 - y_3)^2 + (x_4 - x_3)^2}} = \frac{x_2 - x_1}{x_4 - x_3} ?$$

$$\frac{h_1^2}{h_3^2} = \frac{(y_2 - y_1)^2 + (x_2 - x_1)^2}{(y_4 - y_3)^2 + (x_4 - x_3)^2} = \frac{(x_2 - x_1)^2}{(x_4 - x_3)^2} ?$$

Cross multiplying gives:

$$(x_4 - x_3)^2 \llbracket (y_2 - y_1)^2 + (x_2 - x_1)^2 \rrbracket = (x_2 - x_1)^2 \llbracket (y_4 - y_3)^2 + (x_4 - x_3)^2 \rrbracket$$

Distributing produces:

$$\llbracket (x_4 - x_3)^2 (y_2 - y_1)^2 \rrbracket + \llbracket (x_4 - x_3)^2 (x_2 - x_1)^2 \rrbracket = \llbracket (x_2 - x_1)^2 (y_4 - y_3)^2 + (x_2 - x_1)^2 (x_4 - x_3)^2 \rrbracket$$

Noticing that $(x_2 - x_1)^2 (x_4 - x_3)^2$ appears on both sides of the equation, it can be subtracted from each side leaving:

$$\llbracket (x_4 - x_3)^2 (y_2 - y_1)^2 \rrbracket = \llbracket (x_2 - x_1)^2 (y_4 - y_3)^2 \rrbracket$$

Dividing through first by $(y_4 - y_3)^2$ then by $(x_4 - x_3)^2$ results in:

$$\frac{(y_2 - y_1)^2}{(y_4 - y_3)^2} = \frac{(x_2 - x_1)^2}{(x_4 - x_3)^2}$$

Taking the square root of both sides answers our question positively.

$$\frac{h_1}{h_3} = \frac{(y_2 - y_1)}{(y_4 - y_3)} = \frac{(x_2 - x_1)}{(x_4 - x_3)}$$

Therefore, we can say that the hypotenuse of each right triangle is in the same proportion to each other as the sides are to one another. This further establishes the fact that the slope of a line is immutable no matter which two points on that line are used to determine its slope and introduces the concept of similar triangles using only algebra and the Pythagorean Theorem.

With the same approach, using only algebra, one may set the general form of a line which is tangent to a function equal to that function and solve for the slope of the line [8]. The slope of the line tangent to a function is actually the derivative of the function. Therefore, a student with only an algebra background can acquire a grasp of the concept of derivatives without delving into the sometimes confusing notion of a limit.

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