SOME muMATH ACTIVITIES WITH LARGE CLASSES OF DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

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INTRODUCTION: There has been much attention given to the uses of computer algebra systems (henceforth abbreviated CASs) in undergraduate instruction. However, the bulk of it seems to have been focused at smaller schools where class sizes permit personal interaction with students. Some of these schools even have Honor Systems which free instructors from such mundane worries as plagiarism. Thus, activities designed for smaller schools are often not very helpful to those of us in large state universities where classes of 50 or more are common. In addition to large class sizes, the demands of research are usually heavy in big institutions (and small alike). Thus, left to themselves, many professors will not require graded computer assignments if it takes very much of their time.

This paper will discuss an attempt to respond to these problems by examining some computer assignments the author has incorporated into our linear algebra and differential equations courses.

We assume without apology that undergraduate mathematics students should be exposed to the power and utility of microcomputers in general and of CASs in particular as part of their education; but beyond just witnessing their power, students should see examples where CASs are integrated into the solution of a problem. We have tried to find ways to accomplish this in the face of the constraints mentioned above.

In summary we wanted each exercise to meet the following demands: It should (1) require the solution of a problem relevant to the course, (2) be impractical to solve without a CAS, (3) require very little of the instructor's time and (4) make plagiarism difficult.

FACILITIES: The setting for the exercises is the Mathematics Learning Resource Center (MLRC), a modern facility utilizing electronic technology and conventional tutoring to supplement the mathematics curriculum. Included in the MLRC are 40 networked Texas Instruments Professional microcomputers and 12 networked Texas Instruments Business Pro microcomputers. The MLRC is open 59 hours per week and has a staff on duty to check out software, take up assignments and assist students. This relieves the classroom instructor of some of the burdens of dealing with computer assignments.

A wide range of software is available at the MLRC including the CAS muMATH. It is relatively cheap and runs on any MS-DOS machine. Some of our graduate students and faculty (including the author) have created functions and programs in muMATH's lisp language muSIMP, along with documentation, that enhance the muMATH environment rendering it easier for our students to use. (I regret that I cannot provide examples of muSIMP code here due to space limitations.) In addition to programs that perform each of the three elementary row operations, are the following:

- ROWRED(M); puts a given matrix M into its row reduced echelon form.
- CHR(M); gives the characteristic polynomial of a matrix M.
- RATROOTS(P); searches a given polynomial P for rational roots.
- TSP(M); gives the transpose of a matrix M.
- Y‘; gives $\frac{dY}{dx}$.

Just as importantly, the author has written programs in muSIMP that facilitate the generating of numerous distinct exercises, along with solutions, that are provided to the instructor. Since each student's problem is different, plagiarism is deterred. Of course a student could ask a friend to help him in the computer lab, but he is at least being exposed to the technology and it has been our experience that slight differences in the exercises are enough to prevent deliberate copying. A brief help sheet is also provided that explains muMATH just to the extent necessary to solve the given problem. This frees the instructor from any extra effort other than simply matching up the problem with the solution.

This problem generating capability and the above enhancements make the following activities feasible for large classes.

Linear Algebra Exercises

<u>Row reduction</u>: One very simple exercise is to require each student to row reduce a matrix of size, say, 5 by 8 with non-trivial rational entries. This problem is one that the student will be familiar with but would be quite tedious to solve without a computer and liable to unacceptable round-off error without a CAS.

Constructing a list of examples and solutions is easy to accomplish: For example, begin with a 5 by 8 matrix A in reduced form and a 5 by 5 invertible matrix B. Let's say that column two of A is $[1, 0, 0, 0, 0]^t$ (V^t is the transpose of V). We employ a simple muSIMP program that prints the product B.A, increments the $(1, 2)$ entry of A and repeats. In this way we generate a list of complicated looking matrices which are alike except for the second column, and the solutions are easy to check using the original matrix A. For greater variety and complexity one may of course change A and B.

<u>Diagonalization</u>: Another good exercise is to ask a student to diagonalize a large matrix A, say 6 by 6, with non-trivial entries. Using muMATH it is quite easy to produce a variety of matrices A with integer eigenvalues and corresponding matrices P such that $P^{-1}AP$ is diagonal: One defines a diagonal matrix D and a matrix P each with one variable entry and uses muMATH to compute PDP^{-1}. Invoking the EVSUB command, one may substitute

several values for the two variables in this expression, thus generating a list of desired examples with the accompanying eigenvalues, characteristic polynomials and diagonalizing matrices P.

muMATH may require a long time to compute PDP^{-1} with two variables. One way to speed things along is to define a function, say QUICK(R,S), that substitutes given values R and S for the variables in D and P respectively and then evaluates PDP^{-1}. Each invocation of QUICK takes very little time. If a class is taught in a large section of 100 or more, it is probably not necessary that each student have a unique problem. Twenty different matrices will probably suffice to discourage plagiarism if they are properly distributed.

To solve the problem the student invokes DET(xI - A) or CHR(A) to obtain the characteristic polynomial of A. Instructors have a choice of not divulging the existence of CHR or RATROOTS. Some are content to allow their use while others prefer the student to invoke DET and do a search for rational roots, using muMATH to help out.

Finding a matrix P such that $P^{-1}AP$ is diagonal is a tractable problem thanks to ROWRED. After invoking ROWRED(λI - A) for each eigenvalue λ the student reads the corresponding eigenvectors off the row reduced form of λI - A.

Finally we may insist that the student check his work by using muMATH to calculate $D = P^{-1}AP$. If printers are available hard copies of the results may be made, but hand written copies are not too burdensome.

<u>Roots of Matrices</u>: As an augmentation of the last exercise, the given matrix A has eigenvalues that are perfect cubes and it is required to find a cube root of A; i.e., a matrix B such that $B^3 = A$. Of course the heart of the problem is the above diagonalization procedure followed by $PD^{\frac{1}{3}}P^{-1}$. This is a very nice problem, for just as with the diagonalization exercise above, it uses a great deal of what the student has learned and would be virtually impossible to solve without *both* the theory he has learned and the CAS. I have also found that many students are fairly impressed by two aspects of it: First, when I show them the naive approach of letting B be a matrix with 36 variables and trying to solve the resulting 36 cubic equations in 36 unknowns they are appropriately amused. Secondly, when they are done and check the answer with muMATH by invoking B^3; they are impressed that it actually works and there is no excuse for a wrong answer!

<u>Polynomial curve fitting</u>: In our current text *Elementary Linear Algebra* by Venit and Bishop the method of fitting a polynomial $Q(x)$ to data points $(x_0, y_0), \cdots, (x_m, y_m)$ is to minimize $\sum_{i=0}^{m}(Q(x_i) - y_i)^2$. The text observes that if $n \leq m$ then there is a unique such polynomial $Q(x) = \sum_{i=0}^{n} b_i x^i$; the vector $\mathbf{b} = (b_0, \cdots, b_n)$ satisfies $(U^tU)\mathbf{b} = U^t\mathbf{y}$ where $\mathbf{y} = (y_0, \cdots, y_n)$ and U is the matrix whose i,j entry is x_i^j, $0 \leq i \leq m$, $0 \leq j \leq n$. For examples involving anything other than very small numbers of data points, solving $(U^tU)\mathbf{b} = U^t\mathbf{y}$ would be formidable. With a CAS, it's easy.

Generating lots of examples with solutions is done by a muSIMP program that gives the best fit to m data points by polynomials of every degree from 1 through $m - 1$. One can get several exercises from one such set of computations, by simply incrementing the y-coordinates of the data points. The polynomial solutions are the same except that the constant terms are also incremented.

One may also ask the student submit a graph of the solution if there is such a software package available. Because an elegant check of the answer here is unavailable except in the case of an exact fit by a polynomial of degree $m-1$, such a graph may provide an approximate visual verification. Students often have trouble with this exercise because they do not fully understand the matrix equation $(U^tU)\mathbf{b} = U^t\mathbf{y}$. If they attempt to find the exact fit first and verify it, they may be more likely to get the correct approximations.

Undetermined Coefficient Problems in Differential Equations

A differential operator $L[y]$ is contrived so that its auxiliary polynomial has a desired form; let's say a factor of $(r^2+1)^2$ and one real root, 2, of multiplicity 3. The corresponding exercises ask for solutions to $L[y] = g(x)$ for various functions $g(x)$. Of course muMATH quickly expands $(r-2)^3(r^2+1)^2$ from which we get the operator. In this case, $L[y] = y^{(7)} - 6y^{(6)} + 14y^{(5)} - 20y^{(4)} + 25y^{(3)} - 22y'' + 12y' - 8y$. Letting $g(x) = x\sin x$, $\sin x$, $x\cos x$, $\cos x$, x^3e^{2x}, x^2e^{2x}, xe^{2x}, or e^{2x} gives us eight exercises. To make more exercises one may change the real root "2" to other values. There is an answer sheet generated by a muSIMP program that contains solutions for all the problems. Space does not permit elaboration.

To solve his problem the student is directed to use the method of undetermined coefficients. Thus the student must solve the homogeneous case and guess the appropriate form of a particular solution y_p with variable coefficients. The truly tedious part of the problem now appears when $L[y_p]$ must be computed and this is where the CAS takes over. On the 520K TI Business Pros it takes 3 minutes to evaluate $L[y_p]$ where $y_p = (Ax^6 + Bx^5 + Cx^4)e^{2x}$. After it is done, the equating of like coefficients produces a system of n equations in n unknowns with $n \leq 4$, depending on $g(x)$. This may be solved by using the CAS to invert the coefficient matrix or by hand, but using the CAS as a calculator. Finally, the student is required to check the final answer with the CAS.

Each of the problems presented here requires the student to integrate the CAS into the solution in an essential way, and it demands more than mindlessly pushing some buttons. A principle or method of solution must be understood and some of the work carried out by the student.