RUBIK ALGEBRA

Charles G. Fleming and Judy D. Halchin

Rubik Algebra is primarily a tool for illustrating a variety of ideas from elementary group theory, using Rubik's cube. It is also very useful as a tool for someone trying to discover how to solve the cube. We will first describe the basic capabilities of the program and then a few short examples of how it can be used as an aid in a group theory course.

Section 1. Program features.

When the program begins, the screen is divided into two sections. On the left-hand two thirds of the screen is a picture of Rubik's cube, and the right-hand third contains a menu. This basic screen arrangement is maintained throughout the program, except when the user chooses to view the cycle decomposition of a move and when, on a few occasions, the picture of the cube is temporarily replaced by information the user has requested to see. Menu items are chosen either by using the arrow keys to move a highlighting bar to the correct choice and then pressing the enter key or by just typing the first letter of a menu choice.

The first choice the user may pick from the menu is to perform a sequence of face rotations. When this choice is made, the menu is replaced by a message asking the user to type in a rotation or sequence of rotations. The notation used for describing rotations is one commonly used in books about Rubik's cube. The six faces are called front, back, left, right, up, and down. The first letter of any of these six names denotes a 90 degree clockwise rotation of that face. A minus sign before the letter denotes a 90 degree counter-clockwise rotation of the indicated face. Rotations may be strung together to form a sequence, and numbers may be used to indicate that certain rotations should be repeated. For example, the sequence "r-f2b" indicates a 90 degree clockwise rotation of the right face followed by a 90 degree counter-clockwise rotation of the front face and a 180 degree rotation (two 90 degree clockwise rotations) of the back face. ("Clockwise" on the back face, for instance, assumes the viewer is standing behind the cube, looking at the back face.) A help screen is available at this point in the program to explain this notation for the new user. After entering a sequence of rotations in this form, the user is given the opportunity to indicate the number of times the sequence should be performed. The picture of the cube is then updated to show the cube after the sequence of moves has taken place the indicated number of times. The user can choose an option of turning the entire cube right, left, up, or down to see different sides of it.

The most useful feature of the program, from the point of view of algebra, is its ability to decompose a sequence of face rotations into cycles. By choosing "cycle decomposition" on the menu, the user can see a list of the cycles that comprise the most recent sequence performed. The notation for displaying the cycles is similar to that used for describing rotations. Corner positions on the cube are described by the three faces that meet there. Thus, rfu indicates the corner where the right, front, and up faces meet. Positions in the center of an edge are described by a pair of letters, such as br for the back, right edge. A cycle can then be given as a list of corners or edges. For example, (rf bl) indicates a two-cycle in which the the cubie (one of the twenty-six little blocks that make up the cube) at the right, front edge is moved to the back, left edge position, and the cubie in that position moves to the right, front edge. It also tells us how the cubies will be oriented. Matching up the first faces listed in "rf" and "bl", we see that the right face of the cubie in the right, front position will move to the back face; likewise, the front face will move to the left side. If the cycle had been given as (fr bl), we would have known that the front face of the front, right edge cubie would move to the back. The program also provides a help screen here to explain these notational conventions.

Another useful feature that can be chosen from the menu is the use of a library of moves. Upon making this choice, the user is shown a secondary menu for the use and updating of a collection of face rotation sequences stored in a disk file. The user can choose to add the most recently performed sequence to the file, display the sequences currently in the file, delete sequences from the file, or perform one of the sequences in the file. This feature is useful both for a user wanting to unscramble the cube and needing to save useful sequences, and for an instructor needing to save sequences to be used in a classroom demonstration.

Other options on the main menu include a random scrambling of the cube, unscrambling the cube to restore its original condition, and an option to take back the most recent operation performed. With this last option, a user trying to unscramble the cube can easily undo a wrong move or any number of wrong moves. After this choice has been made, a new menu choice appears--that of restoring the move just taken back. Thus, having made a series of moves, the user can move backward and forward through the series of resulting positions at will. When a new move is finally entered at any point, the ability to restore "taken back" moves is then lost.

Section 2. Applications.

Some of the topics from group theory normally discussed in a first course on abstract algebra are (a) the decomposition of permutations into disjoint cycles, (b) even and odd permutations, (c) the order of an element in a group and, in particular, that the order of an element of a permutation group is the least common multiple of the lengths of its cycles, and (d) conjugation and the effect of conjugation on the cycle structure of a permutation. We will briefly mention how Rubik Algebra can be used to investigate and motivate these topics.

Of course one can calculate the cycle decompostition of a process by hand; however, for even a simple process such as rfu, this is tedius to do. For rfu, the cycle decomposition turns out to be (flu luf ufl), (fru ufr ruf), (dfr frd rdf), (dfl blu rbu rdb), (fu fr uf rf), (df lf ul ub ur br dr fd fl lu bu ru rb rd). One quickly tires of doing hand calculations when permuting the eight corners and twelve edges of Rubik's cube and also keeping track of orientations. Also, there are quite a few processes (elements of the cube group) that one wants to investigate. Rubik Algebra can take any process one wants to investigate and effortlessly generate its cycle decomposition.

Once cycle decompositions are available, a number of concepts and results can be illustrated. By looking at the cycle decomposition of f (or any other single face rotation), we see that f is an even permutation, and, since every element of the cube group is generated from face rotations, we find that all of the processes that can be applied to Rubik's cube are even. A consequence of this is the impossibility of swapping exactly two corners of Rubik's cube while leaving the edges fixed. Such a process would be a transposition on the two corners (an odd permutation) and the identity on the edges (an even permutation), and thus the process would be odd.

Another result that can be illustrated is the fact that the order of an element is the least common multiple of the lengths of its cycles. For example, by looking at the cycle decompostition of rfu, we find cycles of length three, four and fourteen, so the order of rfu is eighty-four. The student can apply rfu to the unscrambled cube eighty-three times (by choosing the repetition factor of eighty-three) and then apply rfu once more and see that the cube has returned to its initial state.

To illustrate the fact that the order of each element is finite, we ask the student to find a single process and the number of times that process must be applied to unscramble a scrambled cube. If P was the process used to scramble the cube, we use the program to obtain the cycle decomposition of P. From the cycle decomposition, we calculate the order of P. If the order is N, then applying the process P N-1 times will unscramble the cube.

The cycle decomposition of an element can also lead us to other interesting elements of the group. For example, the cycle decomposition of the commutator P = rf-r-f is (flu rfu ufl urf luf fur), (dfr drb frd rbd rdf bdr), (fu dr fr). P has order six, so 6P is uninteresting. However, we see that 3P is an interesting process since it has no edge cycles. In fact, the decomposition of 3P is (rfu luf), (drb rdf).

Finally, we mention how to use Rubik Algebra to illustrate the result that conjugation does not affect cycle structure. The process P = l2d-l-f2dfu-f2dfl2d-l-u has for its cycle decomposition (flu luf ufl), (fru ruf ufr), so the result of P is a clockwise twist of the flu corner and a counterclockwise twist of the fru corner. If we wish to apply this twisting action to the dfr and bdr corners, we can do this by conjugating the process P. The process -2f2rP-2r2f moves the dfr and bdr corners into the flu and fru corners, applies the twist, then move the two corners back to their original locations. This gives us a visual check that the cycle structure of -2f2rP-2r2f is the same as for P. We can of course also verify this fact by using Rubik Algebra to calculate the new cycle decomposition. The decomposition for -2f2rP-2r2f is (dfr frd rdf), (bdr rbd drb).

These examples show some of the ideas that can be easily illustrated to students in an elementary group theory course, using the program described here. Such demonstrations, much too lengthy to do by hand, are quick and easy when the proper tool is available and can make effective and memorable illustrations of fundamental ideas.