Changes in Pedagogy and Testing When Using Technologies in College-Level Mathematics Courses [1]

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One of my favorite cartoons shows a student sitting at a table with four calculators on it. A teacher, while bending over the table and touching one of the calculators, is saying "If you have four calculators and I take one away, how many are left?" I like this cartoon for two reasons. One is that for many years we have been urging teachers to use physically manipulable materials to help students to learn and to do mathematics. When thought of in this way, the teacher in the cartoon is effectively using the calculators as manipulatives to help the student to understand and solve the open number sentence $4 - 1 = ?$. This cartoon also represents to me the belief held by some persons that this is the only way to use calculators and computers in mathematics instruction, while at the same time the cartoon shows how ridiculous and ineffective the use of electronic techniologies will be if we permit students to use them only for trivial, sometimes unimportant, tasks.

At present, we do not know a great deal about the effective use of electronic technologies in mathematics. But I opine that as we better define effective ways to use present and emerging electronic technologies in mathematics instruction, we will discover that we must also change the ways in which we teach (i.e., change our pedagogy) and how and what we test. This is probably especially true in mathematics courses taught at the college-level because thus far we have used calculators and computers mostly in relatively trivial ways when compared to the mathematics we are teaching and expect students to learn. In this paper I will discuss some of the ways in which pedagogy and testing change when technologies are used in college-level mathematics courses. However, before I do this I will discuss the issue of *effective use* of calculators and computers in mathematics instruction.

Their Minds Will Turn to Mush!

At present at the school (i.e., pre-college) level, the greatest concern about the use of electronic technologies in mathematics seems to be about the use of calculators. This may be true because calculators are largely regarded as tools (Taylor, 1980) while computers are regarded either as a technology that engages students in computer-assisted instruction or that students program to solve mathematics problems. The present concern about the use of calculators will undoubtedly broaden to include computers as the use of computers as tools increases.

At the school level, a cry often heard from parents, teachers, and school administrators is that if students use calculators for mathematics, they will become dependent upon them and, as a result, will become even more mathematically illiterate than students are presently; that is, *their* [mathematical] *minds will turn to mush!* Thus, these parents, teachers, and school administrators argue that students should not be permitted to use calculators while learning mathematics. As an aside I would comment that these same persons seem unconcerned that their children and students use calculators in almost every other school subject where they are appropriate including science. But this duality isn't new; when I was a student it was expected that I would use a slide rule in physics and chemistry courses but not in mathematics courses (see Harvey, 1989a).

The argument that student' minds will turn to mush is, at best, a weak one. Hembree and Dessart (1986) identified 79 studies of the effects on students of using calculators in learning mathematics in Grades K – 12. From their meta-analysis of the data from these studies of calculator use, they concluded that:

1. In Grades K – 12 (except Grade 4) students who used calculators in concert with traditional instruction maintained their paper-and-pencil skills without apparent harm.
2. The use of calculators in testing produced much higher basic operation and problem-solving achievement scores than did the use of paper-and-pencil alone. This conclusion held across both grade and ability levels. The overall better performance of calculator using students on problem solving appeared to be a

[1] This paper is a revision of a paper with the same title given at the First Annual Ohio State University Conference on Technology in Collegiate Mathematics.

result of improved computation and process selection.

3. Students who used calculators had better attitudes toward mathematics and better self-concepts in mathematics than did students who had not used calculators. This conclusion held across both grade and ability levels. (p. 96)

The 1986 National Assessment of Educational Progress (NAEP) mathematics assessment administered test to students in Grades 3, 7, and 11; at each grade level, one sample of students was permitted to use calculators while another sample was not. At each grade level, the students permitted to use calculators outperformed those who were not ($p < .05$). In addition, data from that assessment revealed that Grade 11 students in the upper quartile of mathematics performance used calculators considerably more in five areas than did students in the lower quartile of mathematics achievement (Dossey, Mullis, Lindquist, & Chambers, 1988, pp. 80-81).

This argument also reveals the view of mathematics and, as a result, the mathematics curriculum that the persons making the argument would seem to have: mathematics is a fixed collection of mechanical skills and techniques. At both the school and college levels it is recognized that mechanical skills and techniques may be important when trying to solve problems but that knowledge of them is only a small component of mathematical knowledge, intuition, maturity, and problem-solving ability (The College Board, 1983a, 1985; Commission on Standards for School Mathematics, 1987; Douglas, 1986; National Council of Teachers of Mathematics, 1980; Steen, 1987a). When, however, the skill-and-technique view of mathematics is adopted, then indeed, calculators should not be used by students because their use takes away the need to learn what is then the substance of mathematics. About the only thing that unites those who see mathematics as a collection of mechanical skills and techniques and those who do not is that both groups are agreed that they want students to learn mathematics.

Human history could be a litany of the tools (i.e., technologies) on which we have become dependent. Physically, these technologies include fire, clothing, wheels, steam and internal combustion engines, locomotives, automobiles, and vacuum cleaners. Mathematically, we rely on the theorems proved and the problems solved throughout history; we regard the mathematics that has been generated throughout the ages not only as knowledge but as a set of tools that we should not avoid but should take and use whenever we need them. Electronic technologies and, in particular, calculators need simply to be regarded as useful tools, and as tools, we need to know when to use them effectively and when not to use them. And like all other tools, we need to know when electronic technologies malfunction, and, when they fail, to have them repaired or replace them.

Finally, even though I have discussed here the arguments advanced by persons interested in the mathematics education of school students, I want to remind college and university mathematics faculty that they may be making the same arguments. I know that I have heard college and university faculty making the argument that graphics calculators or computer algebra systems [2] should not be used in collegiate-level mathematics courses because students will not acquire needed skills and techniques. To some it seems unthinkable that sketching the graph of the function is not the final, mentally consolidating activity associated with computing the first and second derivatives of a function, finding the critical points using the first derivative, and checking those points in the second derivative, or that students do not need to integrate by parts the function f defined by

$$f(x) = x\ \ln(x).$$

I am sympathetic to these arguments since I do not know how much my knowledge of integration, for example, depends upon my skill with integration by parts or partial fractions. Continuing with this example, I do suspect that the skills and my practice of them did little to help me understand the concepts of Riemann sum, partition, and indefinite integration or the Fundamental Theorem of Integral Calculus. My indecision only highlights that we must carefuly determine what are effective uses of calculators and computers in both school and college level mathematics.

Effective Uses

Just as some argue that calculators and computers should not be used at all in mathematics, there are others who argue that *any* use of them is appropriate. These advocates believe that no

[2] I shall call computer algebra systems and related systems, such as *MATLAB, symbolic mathematics sytems.*

matter how students use calculators or computers to learn mathematics, their mathematics educations will be improved. One consequence of this argument is that we should give students electronic technologies and encourage them to devise their own ways of using them. The result of following this line or reasoning would be, I believe, a proliferation of the poor practices we already see around us. I have, for example, seen people using calculators to find the sum or product of two one-digit numbers. I have also encountered persons who seem to believe that electronic technologies can solve any problem if they "push the buttons" long enough; some of these were my students who, on tests, spent a lot of time making computations and not enough time in analyzing the problem they have been given to solve or in devising a sensible plan for solving the problem.

As I have stated, we know little about the effective uses of electronic technologies in mathematics and especially, in college mathematics courses. We can best judge the effectiveness of uses by specifying what outcomes we expect. I wholeheartedly subscribe to the outcomes sought by the NCTM's Commission on Standards for School Mathematics (1987) and the Tulane Conference (Douglas, 1986); that is, I believe that we should seek uses that promote improved (a) conceptual learning, (b) problem-solving performance, (c) insight into what mathematics is and how it is generated, (d) mathematical intuition, and (e) attitudes and motivation. Effective uses of calculators and computers can help us achieve these outcomes by correctly *focusing student attention* on higher-order learning instead of low–level skills and techniques, and by *reducing or removing instruction* on skills and techniques and replacing it with instruction for those things we seek – concepts, solved problems, insights, intuition, and enthusiasm for mathematics. Here are some examples that indicate to me my faith is well placed.

One example is the conclusions reached by Hembree and Dessart (1986) that were described. There are four additional examples at the college level.

Heid (1988) studied the effects of the use of graphic tools and symbolic mathematics systems on student understanding in an applied calculus course. During the first 12 weeks of instruction the 39 students in the treatment group used these tools to perform routine manipulations; only during the last three weeks was skill development taught. The students in the experimental treatment showed better understanding of the course content and performed almost as well on a final examination of routine skills as did a class of 100 students who had practiced the skills during the entire 15 weeks.

During 1987-88, Kenelly (unpublished) taught the required introductory calculus syllabus to students who used Hewlett Packard *HP-28C* symbol manipulation calculators. Kenelly reported that the entire syllabus could be covered, that students requested they be taught theorems and their proofs so that they might understand the calculator's symbolic manipulations, and that they were enthusiastic and highly motivated.

The effects of using MACSYMA ^{R} were studied by Palmiter (1986). The experimental group ($n = 40$) studied integration as did her two control groups; the control groups were each comparable in size to the experimental group. The experimental group used MACSYMA while studying the materials; they completed their study of integration in one-half the time required by the control groups. Five weeks and 10 weeks after the end of the instructional treatment, concept and computation tests were given. The test scores of the experimental group was significantly better than those of the control groups on all of the tests. While taking the computation tests, the experimental group used MACSYMA; they completed the test in half the time required by the control groups.

During the Fall Semester, 1988-89, I (Harvey, 1989b) taught a single section of college algebra using *Precalculus Mathematics: A Graphing Approach* (Demana & Waits, 1988) and *Master Grapher* (Waits & Demana, 1986), a graphics tool. The usual syllabus was covered; in addition, considerable time was spent teaching students ways of solving problems using geometric representations generated with the *Master Grapher.* A 25 item algebra test was given as a pre- and posttest to this class of 27 students; there were complete data for 25 of these students. Analysis of those data revealed a statistically significant increase ($p < .001$) in mean achievement from the pretest to the posttest; the mean score on the pretest was 8.76 (s.d. $= 2.85$) while the mean score on the posttest was 13.31 (s.d. $= 3.61$).

The Three Ages of Technology

Since we began to experiment with the use of calculators and computers in mathematics education, there have been three technological ages. During the first age (c. 1965-75), the primary use of computers was to teach students a programming language (e.g., BASIC) and to encourage or require them to write computer programs to solve problems. The beginning of the second age coincided with the introduction of both hand-held calculators and microcomputers. During this age, from 1975 to 1985, students may still have been expected to write computer and calculator programs but the primary activities were computer-assisted tutorials and the uninstructed use of calculators.

Because of the ways in which calculators and computers were used during the first two ages, few, if any, changes in mathematics pedagogy were needed. Since students were left to their own devices, expected to write programs to solve the problems outside of class, or to interact passively – again outside of class – to computer assisted tutorials. Together these two ages comprise the passive pedagogic period. During this period mathematics teachers went along doing the same thing they had always done in their classrooms; they used few of these technological innovations on a regular basis in their instruction.

The present age is one in which we are teaching students to use technological tools and in which we are actively using those same tools in classroom instruction on a day-to-day basis. For these reasons, I call this the active pedagogical period. I am not quite certain why mathematics teachers, and especially college mathematics faculty, have moved from a passive to an active stance in their use of technologies.

One reason may be that we have recognized that students may not or cannot discover the appropriate uses of these tools by themselves. My own observation of students using scientific calculators, for example, is that they need to know more about the keys that invoke the built-in mathematical functions and parentheses.

A second reason may be that we have discovered that technological tools can greatly assist us to teach. For example, using *Master Grapher* (Waits & Demana, 1986), I can quickly and easily look at the graph of a function both globally and locally. In addition, the graphs drawn by this graphing tool are better drawn and more accurate than those that I can sketch.

A third, and possibly penultimate, reason may be that they make mathematics more enjoyable while also teaching us new things about mathematics that we thought we had learned and learned well. For example, the graphing tools (e.g., a Sharp *EL - 5200*) have permitted me to explore geometrically functions that I had never considered before like $f(x) = x \ln(x) - \sin(x)$ and to see graphically that (a) f has a relative minimum (at $x \simeq 0.76$), (b) the first derivative is zero and the second derivative is positive at this point, and (c) that f has an infinite set of inflection points. (The graph of $f''(x) = 1/x + \sin(x)$ shows more clearly that f has that infinite set of inflection points than does the graph of f.) Figure 1 shows that graphs of f, f', and f''.

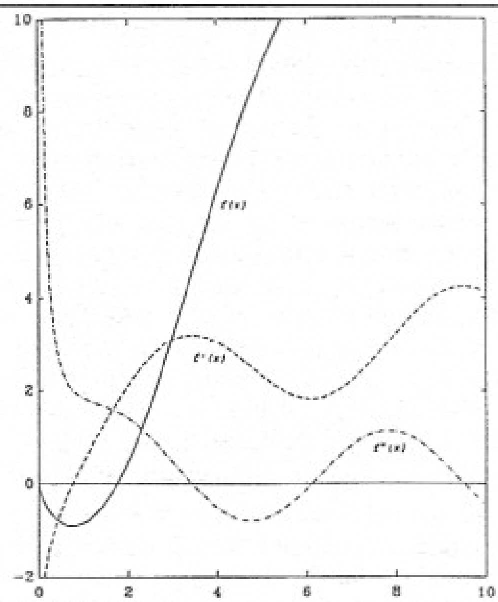

Figure 1. Graphs of f, f', and f'' where $f(x) = x \ln x - \sin x$

Fourth, it may be because these tools have become easier to use and to learn to use and have capabilities that were previously not generally available. One can learn to graph functions using a graphing calculator in less than an hour; at that point, one is ready to begin learning how these calculators can be effectively used in mathematics. Microcomputer symbolic mathematics systems have capabilities only possessed by large, mainframe computers 10 years ago.

What Choices Are There And What Are the Pedagogical Consequences?

At present when considering a choice of technology tools for college-level mathematics courses, there are three choices: programmable and non-programmable scientific calculators, graphing calculators and computer graphing tools, and computer symbolic mathematics systems. The changes in pedagogy that can or should occur when these technologies are used grows as the capability of the tool increases. Thus, the changes in pedagogy that I will discuss when using scientific calculators will also apply to graphing tools and to symbolic mathematics systems.

Scientific Calculators. Among the three tools, scientific calculators are the most widely accepted and the most widely used; a recent survey of 477 college faculty at 183 colleges showed that this is true (Kupin & Whittington, 1988). Table 1 shows the number of respondents who teach each of seven courses and the percent of those who make some use of calculators in those courses.

Scientific calculators have been available longer and are relatively inexpensive (varying in price from about $12 to about $60). When first introduced some college faculty quickly required or permitted their use while other faculty barred them from their mathematics classes. However, scientific calculators–programmable and nonprogrammable–are now familiar equipment in many precalculus and calculus classes. For example, at the University of Wisconsin-Madison we include in our timetable a statement that students taking introductory calculus courses are expected to have a scientific calculator. Some of the uses of scientific calculators in mathematics are these:

1. to facilitiate numerical estimation, approximation, and computation (including numerical differentiation and integration),
2. to make tables so as to search for patterns, to help students sketch graphs, and so forth, and
3. to check results (i.e., answers).

Table 1

Calculator-Use in College Level Courses

Courses	Number of Respondents	Percent of Respondents Indicating "Some Use"
College Algebra	212	77%
Precalculus	274	85%
Statistics		
Non-Calculus	206	94%
Calculus	58	97%
Calculus	426	84%
Discrete Math	140	62%
Linear Algebra	165	81%

Adapted from Kupin & Whittington (1988).

One respondent to the survey conducted by Kupin and Whittington (1988a) may have best summed it up by saying "any topic with calculator-friendly algorithms."

If we actively use scientific calculators, the pedagogic implications of these uses of calculators would seem to be these.

- *We must let students use their calculators as often as they like.* This means, for example, that we can no longer say to students, "You can use your calculators while you solve this set of problems (e.g., you homework) but you cannot use them while you solve that set of problems (e.g., the test problems)."
- *We need to describe and discuss with students the situations in which calculator use is and is not appropriate.* For example, an appropriate use would be the development of a table to graph $f(x) = x \ln(x) - \sin(x)$ in the interval $(0, 5]$ while an inappropriate use would be to use a numerical integration program to find the definite integral of $f(x) = 1/x^2$ in the interval $[1, 5]$. We presently practice this policy with the skills and techniques we teach; thus we simply must extend our policy to include calculators (and computers). If we regularly use a tool in preparing for and teaching our courses, then we will be better able to easily describe appropriate and inappropriate uses.
- *We need explicitly to show students the kind of thinking and planning needed before calculator or computer use begins and after it is concluded just as we do presently with skills and techniques.* By extending this kind of instruction to calculator and computer use we will have additional opportunities to (a) teach problem-solving heuristics and strategies, (b) help students better to understand underlying concepts and principles, (c) give students instruction on estimating and approximating answers, and (d) help students better discern correct from incorrect results.
- *We need consistently to use calculators, both in and outside of the classroom, to show students that calculator and computer use in mathematics is appropriate, important, and acceptable.* I believe that many college students continue to believe that it is acceptable to use a calculator in, say, physics courses but that its use in mathematics courses is unacceptable just as I believed as a student that slide rules were unacceptable. This belief

makes students, in a way, calculator and computer phobic and so, afraid that their knowledge of mathematics will not grow appropriately or be adequate if they use these tools while learning mathematics. In other words, they suffer from a sort of self-inflicted "My mind will turn to mush!" argument.

Graphing Tools and Symbolic Mathematics Systems. The use of graphing tools and symbolic mathematics systems is newer, more arguable, and less well explored. In a great part of the undergraduate mathematics curriculum these tools could change what students learn, how they learn it, and how we teach it to them. Many argue that if we fail to include the use of graphing tools and symbolic mathematics systems into the way we teach mathematics the result may be a "Latinized" mathematics curriculum that is studied only by those preparing to be mathematicians (i.e., mathematics scholars); these students would essentially be studying a dead language as far as the rest of the world, including most of the academic world, is concerned (Osborne, of these *Proceedings*; Steen, 1987b) The advantages of using calculators and computers in parts of the college mathematics curriculum have been discussed (Douglas, 1986, pp. vii-xxi; NCTM, 1988; Steen, 1987c; Tucker, 1987; Zorn, 1986; Zorn, 1987). Others have expressed concern or have urged caution in changing the undergraduate mathematics curriculum (Buck, 1987; Gillman, 1987; Stein, 1986). We need to proceed carefully, though expeditiously, in incorporating these tools. Here are some suggestions.

- *We need to analyze carefully the content that we presently teach and that we would like to teach.* That is, we need to carefully reexamine what it is that we want students to learn and to compare that to what we presently teach them and to what we think they *can* learn. Most of the discussions about the inadequacies of present calculus courses (Douglas, 1986; Steen, 1987a) have identified that our present courses teach too many skills and techniques and too few concepts and too little problem solving. My colleagues at the University of Wisconsin-Madison and at the other UW-System institutions acknowledge this is true and that we would like to improve our teaching of concepts and problem solving. However, we also feel that our present calculus courses are very challenging for the majority of our students. The number of students who do not successfully complete introductory calculus courses on our campuses is about 25% (University of Wisconsin-Madison, Department of Mathematics Calculus Committee, unpublished; Harold Schlais, personal communication); these rates are consistent with other estimates (Douglas, 1986, p. xvii). Only a careful analysis of what it is essential for students to learn coupled with an analysis of (a) what we think students can learn and when they can learn it and (b) which students we want to include and exclude from our mathematics courses will tell us how to improve our undergraduate mathematics curricula. These decisions need to be made locally since they depend upon many factors including the size and admissions requirements of the institution, the qualifications and intended majors of the institution's students, the availability of calculators, computers, and software, the technological literacy of faculty (and teaching assistants), and the willingness and ability of faculty (and teaching assistants), and the willingness and ability of faculty and the institution's administration to change.
- *Once the content of the mathematics curriculum has been examined, we need to determine the ways that particular tools can help us to teach that content.* Undoubtedly, this step is not one that occurs after all of the decisions have been reached about the content of the curriculum, but I have placed it second in my list to emphasize my belief that the choice of appropriate mathematical content is most important (Harvey, 1989a).
- *We must not cling to our present ways of teaching.* There are at least two things that must change; both are related to my earlier observation that to be most effective our instruction will need to include regular use of the tools we select.

  First, it seems we will need to give up our roles as expositors, leaders, and "the sources of knowledge" and become instead resource persons to and, on occasion, co-learners with our students. Let me describe a typical session in my algebra class using *Precalculus Mathematics: A Graphing Approach* (Demana & Waits, 1988) and *Master Grapher* (Waits & Demana, 1986). My classroom is equipped with a computer, screen, and color projector. During most class sessions I usually talk for a few minutes about the algebraic techniques that the students are learning and the ways in which they can and should be used; a majority of the class time is spent engaged in problem solving using both algebraic and geometric techniques. During problem solving I basically ask questions, make suggestions, and operate the computer while students supply me with the problems to be solved, the questions to be answered, the procedures to be used, and most of the answers. High school mathematics teachers who are participating in the 1988-89 field test of *Precalculus Mathematics* reported that their classes proceed in about the same way (Bert Waits, personal communication).

Second, we must be willing to rearrange the order in which topic and ideas are presented. Let me relate two experiences here. One comes from my present college algebra course; the other from John Kenelly's experiences in teaching calculus with the Hewlett Packard *HP-28C* calculators.

A typical way of teaching college algebra students to find the real zeros of a polynomial function f of degree three or higher is:

1. Check to see if f has a zero at $0, 1$, or -1 by directly computing the value of f at those points.
2. Estimate the number of real zeros using Descartes' Rule of Signs.
3. Use the Rational Root Theorem to make a list of the possible rational zeros of f.
4. If Step 2 has shown there are no positive or no negative real zeros, delete the appropriate entries from the list developed in Step 3.
5. Check the values remaining in the list by directly computing their images under f to see if they are zero.
6. Use the information gained in the first five steps to factor f as completely as possible, and in most cases, *stop* whether or not the polynomial has been completely factored.

When using a graphing tool polynomials of degree three or higher may not be factored at all. Instead a complete graph (i.e., one that shows the global behavior and as much of the local behavior of the function as possible) is drawn; then, using this graph and successive graphs of local parts of the function, approximations to *all* of the zeroes of the function are obtained. However, since it remains important that students find exact solutions, in this case rational zeros, to some problems the next procedure also is used. Steps 1, 2, and 3 of the typical method outlined previously are followed. Using the information obtained, a graph of the function is drawn using a graphing tool so that a portion of the domain that contains all of the remaining possible rational zeros is shown. The graph is inspected, and more of the rational zero candidates are eliminated. The remaining candidates are tested by directly computing the value of f at those points. If, at that point, the irrational real zeros of f are desired, then a complete graph is drawn, an the irrational roots are approximated.

This example points out how graphing rearranges the order and the way in which algebraic techniques are used. Graphing has become an integral part of the problem-solving process instead of an end result of it.

John Kenelly (unpublished) reported similar instances in which the content of the typical calculus course was rearranged because of the use of the *HP-28C* in his class; two of the instances are notable. In the first instance he reported that for three decades C students in his classes had been unable to apply the chain rule to three-step problems. When teaching the chain rule to his *HP-28C* class he and his students worked a number of chain rule problems with the sequential differentiation key on the calculator. He reported that on the examinations a number of C students in the experimental section were able to work three and four-step chain rule problems successfully.

Kenelly also reported that graphing was an important tool in his experimental calculus class. "The unit [the *HP-28C*'s] turned around the way that the students looked at graphing and the calculus. The class started out with the graph and used the calculus to understand the process – not the historically opposite direction. This was one of the most successful uses. In fact, the students simply could not fathom that people had to use such long and involved calculations of derivatives just to find the general description of the curve."

- *Using graphing tools and symbolic mathematics systems will enable us to emphasize the liason between geometric, algebraic, and analytical thinking.* In many instances use of these technologies can help establish this liaison because this problem-solving strategy can be applied:

1. find an appropriate algebraic or analytical representation of the problem,
2. use a graphing tool or symbolic mathematics system to develop a geometric representation of the problem or to manipulate the originally derived representation,
3. obtain an (approximation to the) answer from the geometric representation or from the symbolic manipulation, and
4. prove that the answer obtained is correct or is a good one.

- *We should use these technologies to teach students about the global and local behavior of systems and of the interactions between them.* As mathematicians we can predict the behavior of specific functions and relations because we know the class of functions to which they belong and have in our mind's eye a picture of the graphs of the funtions in that class. As a result we usually only need to experiment a little to discover how particular functions differ from our generic images. Students seem not to learn very well that similar functions have similar graphs and, sometimes, vice versa. For example, when you show students the funtion $f(x) = x^2$ and a graph of it and then ask those students what, in general, a graph of $g(x) = 2x^2 - x + 3$ might look like, they often have no idea.

With a graphing tool it is easy to see that the leading term of a polynomial function describes the behavior of the function when $|x|$ is large by showing students that, in an appropriate viewing rectangle as shown in Figure 2, the global behavior of two polynomials of the same degree are quite similar. I have no quantitative data to support this claim, but I firmly believe that the students in my experimental college algebra section have come to know that all quadratic polynomial function are parabolas, that all third-degree polynomial functions have graphs that resemble that of $f(x) = x^3$, and so forth.

At the same time a graphing tool can help students to understand the remaining part of the theorem from theory of equations that I have started to recite; namely, that when $|x|$ is small then terms of lower degree in a polynomial function begin to dictate its behavior. Figure 3 shows another graph of the function $f(x) = x^3 - 4x^2 + 4x$. In the interval from $x = 0$ to $x = 2$, it is clear that the x^3 term no longer describes the behavior of this function. At that point estimating the relative sizes of x^3 and $4x^2$ shows that the latter term is larger as students come to expect. Discoveries about the local behavior of functions lead naturally to a discussion of local maxima and minima and to some knowledge of where they might be found; as a result students should be better able to estimate answers and to determine correctness of their results.

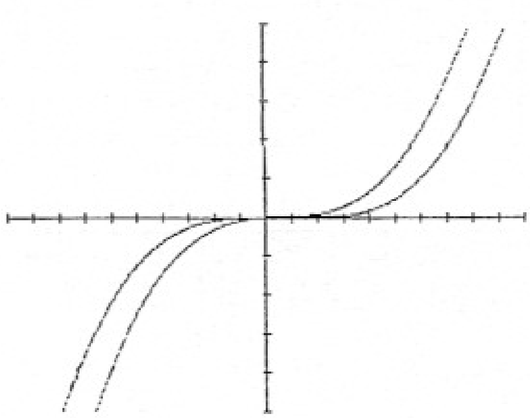

Figure 2. Graphs of $f(x) = x^3$ and $g(x) = x^3 - 4x^2 + 4x$ in the Viewing Rectangle $[-10, 10]$ by $[-500, 500]$

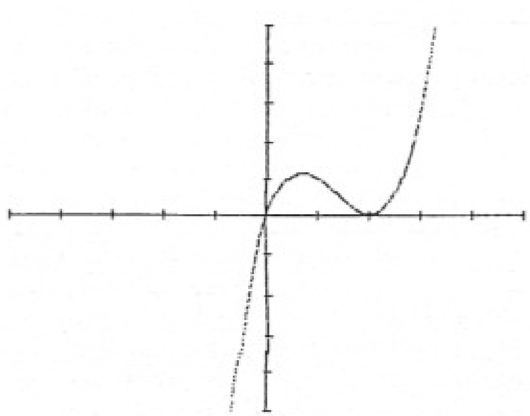

Figure 3. Graph of $g(x) = x^3 - 4x^2 + 4x$ in the Viewing Rectangle $[-5, 5]$ by $[-5, 5]$

Study of local behavior in this way can also lead students to understand results that are deep and could not be taught otherwise – at that time or place. From Figures 4, 5, 6, and 7 you see ample evidence of the piecewise linearity of $f(x) = x^3 - 8x + 1$. Figure 4 shows the local behavior of f; in Figure 5, the part of the graph of the function that will be magnified for consideration is shown in the box. Figûre 6 shows the result of the magnification; Figure 7 shows the result of a subsequent magnification. After students see this over and over again as they approximate zeros, local maxima and minima, and the points of intersection of curves, they come to understand piecewise linear approximation. [3]

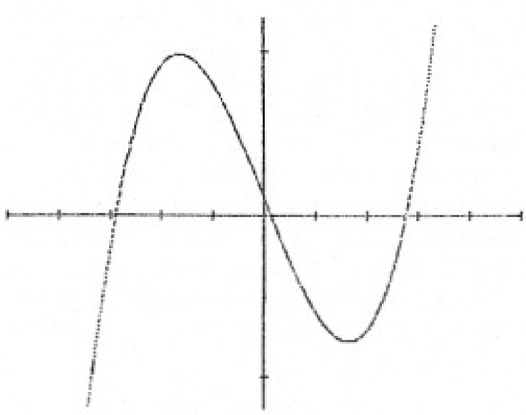

Figure 4. Graph of $f(x) = x^3 - 8x^2 + 1$ in the Viewing Rectangle $[-5, 5]$ by $[-12, 12]$

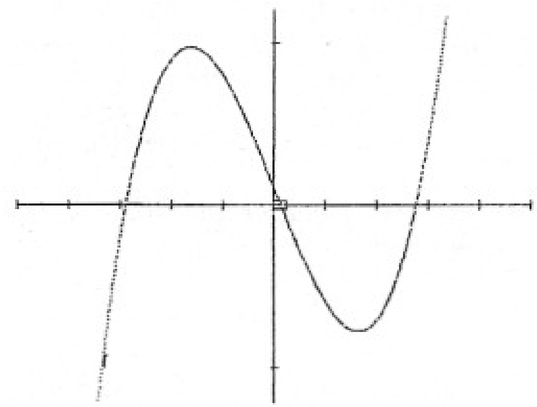

Figure 5. Graph of $f(x) = x^3 - 8x^2 + 1$ in the Viewing Rectangle $[-5, 5]$ by $[-12, 12]$ with the Zoom-in Rectangle Shown

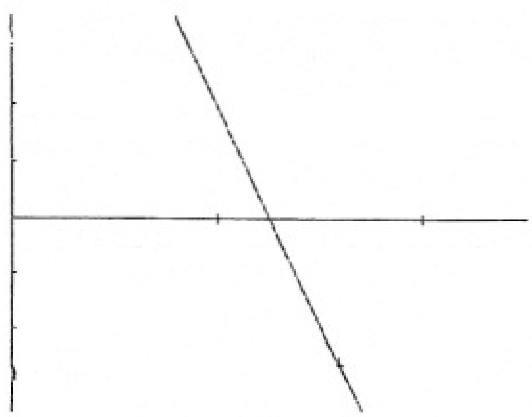

Figure 6. Graph of $f(x) = x^3 - 8x^2 + 1$ in the Viewing Rectangle $[0, 0.25]$ by $[-0.36, 0.36]$

[3] In my classes students came to understand that *Master Grapher* plots points and then connects those points with straight line segments to produce an approximation to the graph of the function being pictured. Thus, when spurious lines and points appear on the scene because an inappropriate viewing rectangle has been used or because the function has a vertical asymptote, students sucessfully sort out the graph of the function from this "garbage."

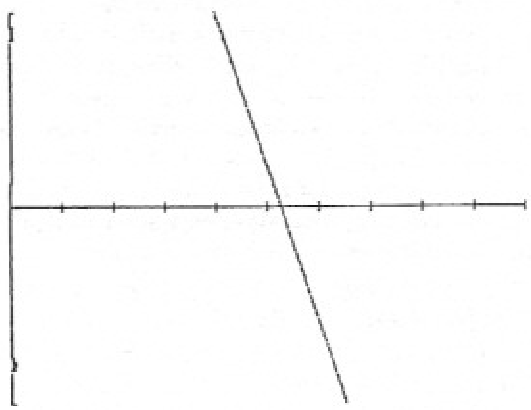

Figure 7. Graph of $f(x) = x^3 - 8x^2 + 1$ in the Viewing Rectangle $[0.12, 0.13]$ by $[-0.01, 0.01]$

Finally, because it is easy to look at related problems, these technologies can also help us to teach students to discover and to generalize. For example, consider the function

$$f(x) = a(x - b)^n + c$$

where n is a positive integer and $a, b,$ and c are real numbers. After beginning with the case of $a = 1$ and $b = c = 0$, it is easy to consider other values of $a, b,$ and c quickly. From this students will generalize the amount and direction that b translates the graph horizontally, the amount and direction that c translates the graph vertically, the way in which a affects the overall shape of the graph, and the rotation in space of the graph about the x-axis when a is negative.

- *Because calculators and computers are discrete machines, both continuous and discrete perspectives must be included.*
- *A part of instruction should be devoted to teaching students how an when to use graphing tools and symbolic mathematics systems.* I realize I have already stated this, but I want to repeat it here because it is especially true for these two technologies because of their added capabilites. The survey by Kupin and Whittington (1988) shows that between 6% and 42% of the respondents that let students use calculators spent "some class time" showing students how to use their calculators and that 26% of the respondents who permitted computer use taught or assisted their students to use them. The amount of time spent was not quantified. The results of this survey indicates that those presently using calculators and computers in their college-level courses recognize that instruction on the use of these technologies is important and possibly, necessary.
- *We will need to spend time with students talking to and working with them while they use graphing tools and symbolic mathematics systems.* This is an obvious suggestion when the uses of technologies are thought of as important problem-solving strategies. It is also important during this period in time when we know so little about effective uses of technologies and about how and how well students will learn to use them. This kind of information could help us to transform the college mathematics curriculum into one that better teaches concepts, problem solving, and applications.

As already stated, the use of calculators, graphing tools, and symbolic mathematics systems cannot be limited to their use by teachers or to student use in class or while doing homework. Effective uses of these technologies by students will mean that they come to regard them as tools that they use much as they presently use pencils and paper. Thus, tests at all levels will have to assume that students will use these tools, and so, test makers will have to design tests with that assumption in mind.

Testing with Calculators and Computers

Undoubtedly, the college mathematics faculty who presently expect their students to have and to use calculators and computers and who permit the use of those tools on tests already recognize that the tests they administer to their students need to be different. Those faculty have discovered that some of the test questions they previously gave to their students no longer are appropriate in that (a) a correct answer obtained by using a calculator or computer does not reveal if the students understand the underlying concept, algorithm, or technique or (b) the difficulty level of the question has been lowered. In addition, these faculty have also found that the use of calculators and computers in their courses permits them to test understandings not possible when only paper and pencils were used, that test questions may be ones that use realistic data or that have good approximations or estimates as answers, and that they have to be careful in developing their tests to generate questions that test mathematics knowledge and not simply students' abilities to manipulate their calculators or computers. I suspect that if this collective knowledge were analyzed and synthesized, we would have a good, possibly definitive, picture of how both test questions and tests can be validly and reliably constructed to be calculator-based or computer-based. Unfortunately, this is not so; our knowledge of calculator- and computer-based mathematics tests is presently "personal" or "fugitive" knowledge.

In 1975 the National Advisory Committee on Mathematics (NACOME) also recognized that calculator use during testing would "most certainly be invalidated in 'calculator classes'" (NACOME, 1975, p. 42). From about that time to the present the National Council of Teachers of Mathematics (NCTM) has urged that calculators be used in teaching and learning mathematics and in 1986 this organization recomended that "test writers integrate the use of the calculator into their mathematics materials at all grade levels" (National Council of Teachers of Mathematics, 1986).

Until recently recommendations like those made by NACOME and NCTM seem to have had little effect at the national level. Until 1986, there had been little experimentation with calculator- or computer-based mathematics tests at this level.[4] In 1983 and 1984, the College Board's Advanced Placement Program permitted the use of calculators on the Advanced Placement (AP) Calculus Examinations. During the time that calculator use was permitted on the AP Calculus Examinations, the test questions were designed to be "calculator-neutral or calculator immune;" that is, the questions were designed so that calculator use neither enhanced or hindered a student's opportunity to solve the problem successfully. This is contrary to the position I have recommended in that it encouraged students to use calculators while taking their AP calculus courses but denied them an opportunity to use their calculators effectively on the test. The reasons the use of calculators on the AP Calculus tests were discontinued were:

> The MSAC [Mathematical Sciences Advisory Committee] recognizes the problems generated by the use of calculators in a calculator-independent situation. The difficulties caused by rapid changes in technology, the lack of equity of access to sophisticated calculators because of expense, as well as administrative and security concerns, are cogent reasons for suspending the use of hand calculators on the AP Calculus Examinations." (Kennelly, 1989b)

The MSAC recognized that there are problems generated when calculators are used in calculator-independent situations. At the same meeting at which they discontinued calculator use on the AP Calculus Examination, the Committee urged that an emphasis be placed on designing tests that allow the use of calculators as aids during testing.

Symposium on Calculators in the Standardized Testing of Mathematics. Because of their common interest in developing mathematics tests requiring the use of calculators the College Board and the MAA jointly sponsored the Symposium on the Use of Calculators in the Standardized Testing of Mathematics in September 1986 (Kennelly, 1989a). Three recommendations from that Symposium are especially relevant here.

1. Mathematics achievement tests should be curriculum based, and no questions should be used on them that measure only calculator skills or techniques.
2. When addressing a particular test item, an important skill is choosing when and when not to use a calculator. Consequently, not all of the questions on a calculator-based mathematics achievement test should require the use of a calculator.
3. There should be no attempt to place an upper limit on the level of sophistication that calculators used on tests should have. Any calculator capable of performing the operations and functions required to solve the problems on a particular examination should be allowed.

While the Symposium participants did not consider or discuss the development of computer-based tests, I believe that these recommendations also extend to the development of those tests as well.

The first recommendation reaffirms and extends to calculators the traditional tenet that the mathematical objectives that are tested should

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[4] The College Board Advanced Placement Examinations in Physics and in Chemistry permit the use of scientific calculators. The National Society of Actuaries permits calculator use on its actuarial examinations. It seems likely that most of the national examining organizations in the sciences and business permit calculator use on their tests.

dictate the kind of questions included on the test. It also asserts that items intended solely to test a student's calculator facility should not be included on mathematics tests. The second recommendation has several valid interpretations. One interpretation is that is is not necessary to give tests consisting only of calculator-based (or computer-based) items though on occasion this may be necessary because not enough calculators or computers are available to test all students at the same time. The second recommendation can also be interpreted as meaning that it is important to test whether students know when and when not to use a calculator. The third recommendation may be naive because in 1986 graphing calculators had just been introduced, the Hewlett Packard *HP-28C* had not yet appeared, and graphing tools and symbolic mathematics systems were only beginning to be explored. If tests and test questions will have to be prepared so that, for example, any calculator can be used then it may be that students who have better calculators and who know how to use them will have an advantage on some test items. As examples let me examine the items shown in Figure 8.

Example 1. Determine the number of real solutions of the equation

$$4x^3 - 10x + 17 = 0.$$

Example 2. Find one real solution for the system

$$y = 4 - x^2 + 3x^3$$
$$y = -1 + 2x.$$

Figure 8. Examples of Mathematics Items On Which Use of Different Calculators Could Produce Inequities

Both examples are very much alike in that a graphing tool makes both of them easier.

When responding to the first item, a student using a scientific calculator without a programmed routine to approximate the solutions could best solve this problem by using Descartes Rule of Signs first to determine that there are two positive real solutions or no positive real solutions and that there is one negative solution. At that point the student would need to sketch the graph of the nonnegative domain values to determine how many positive, real solutions there are; a scientific calculator could be used to develop the table of values for that graph. On the other hand a student who has a graphing calculator would simply develop a complete graph of the function whose zeros are sought and would immediately see that there is only one real, negative solution. A student having a Sharp *EL-5200* calculator could, at that point, press the **[SOLVE]** key and determine that an approximation to the zero is $x \simeq -2.12$; so, changing this question to read "Determine a real solution of ..." would not remove the advantage held by a student having this particular calculator.

The second example asks for an x and a y value that satisfies both equations. Students using scientific calculators would need to follow the steps presently taught: substitute the value of y in the second equation into the first equation, combine like terms, solve $3x^3 - x^2 - 2x + 5 = 0$ for a value of x using the techniques just discussed, and substitute that value into the second equation to find the corresponding value of y. A student having a graphing tool could graph both functions and easily determine the point of intersection using the built-in features of that tool.

Both of these examples show that inequities may occur during testing if students do not have approximately the same technological tools available; the disparities might be even greater if some students have only scientific non-programmable calculators while others have Hewlett Packard *HP-28C*'s. I typically ask students having graphics calculators not to use that facility during tests since other students in my class do not have those calculators. I also give students take home problems that comprise a part of their test so that they each can have the opportunity to work these problems using a graphing tool.

Development of Calculator-Based Tests. Prior to 1986 the Mathematical Association of America's (MAA) Committee on Placement Examinations (COPE) had been discussing the development of calculator-based college-level placement tests; in August 1986, COPE received a grant from Texas Instruments Incorporated to the MAA Calculator-Based Placement Test Program (CBPTP) Project. Development of CBPTP tests began in October 1986. Using the present tests of the MAA Placement Test Program as a starting point, the CBPTP Project test development panels are developing calculator-based placement tests. About 25% of the items on the CBPTP tests will be calculator-active; each CBPTP test will expect students to have a scientific calculator available to them while taking that test. The test panels have been using these definitions of a calculator-active test item and a calculator-based test.

> A *calculator-based* [calculator-active] *test item* (a) is an item containing data that can be usefully explored and manipulated using a calculator and (b) has been designed to facilitate active calculator use.
>
> A *calculator-based mathematics test* is one that (a) tests mathematics objectives, (b) has some calculator-based test items on it, and (c) has no items on it that could have been but are not calculator-based except for items that are better solved using non-calculator based techniques. (Harvey, 1989a)

The first two of the six CBPTP tests have been developed and will be published in 1989; they are the Calculator-Based Arithmetic and Skills Test and the Calculator-Based Calculus Readiness Test. Two additional tests are presently being developed and will be published in 1990; they are the Calculator-Based Basic Algebra Test and the Calculator-Based Algebra Test. The remaining two placement tests will be developed during 1989 and 1990 and will be published in 1991; these two tests are the Calculator-Based Advanced Algebra Test and the Calculator-Based Trigonometry Test. As they are published these tests will be included in the MAA's Placement Test Program test packet. An MAA Note describing the development of these tests and their characteristics will be published in 1991.

The College Board has initiated development of a new version of the Mathematics Level II Achievement Test that will be calculator-based (Harvey, 1989c). This test is presently still in the prototype development stages. There are plans to offer this Math II-C version of the Mathematics Level II Achievement annually when it is completed.

Calculator-Active Test Items. An analysis of two of the MAA Placement Test Program tests showed some of the items were not appropriate for calculator-based tests for two reasons: the items no longer tested the objectives they were intended to test or they tested calculator facility instead of mathematical knowledge. An analysis of the items from the Scholastic Aptitude Tests (SAT) (College Board, 1983b) showed that none of those items were not appropriate for calculator-based tests (Harvey, 1989a). The contrast between the two sets of items is that those on placement tests tend to test the mathematical content (i.e., achievement) of high school mathematics courses and lower-level knowledge while the SAT items measure aptitude instead of achievement. In addition, few of the SAT items were ones requiring much computation and thus, were not calculator-active when judged using a scientific calculator. So, then, what is a calculator-active mathematics test item? Figure 9 gives some examples of calculator-active items drawn from among those developed for the new MAA calculator-based placement tests; the correct answers are marked with asterisks.

Example 1. The approximation of $(1+1/6)^4$ correct to 4 decimal places is
(A) 1.0008 (B) 1.1667 (C) 1.8526 *
(D) 2.1614 (E) 4.6667

Example 2. If $n \times n \times n = 63$, then which of the following is closest to n?
(A) 0.047619 (B) 3.979057 * (C) 21
(D) 189 (E) 250,047

Example 3. If a certain buffalo population increases by a factor of 1.1 every year, then in 15 years it increases by a factor of
(A) 1.500 (B) 3.797 * (C) 4.177
(D) 16.500 (E) 19.666

Example 4. The sequence of number
$(3/2)^4, (4/3)^6, (5/4)^8, \ldots, ((n+1)/n)^{2n}, \ldots$
approaches
(A) 1 (B) 2.179 (C) 6.192
(D) 7.389 * (E) no finite number

Figure 9. Examples of Scientific Calculator-Active Items

The first two examples in Figure 9 were developed for the Calculator-Based Arithmetic and Skills Tests; the third and fourth items will appear on the Calculator-Based Calculus Readiness Test. For the intended student audience each item is one that could not be given if students were not permitted to use scientific calculator. The first two examples test student understanding of exponentiation; Example 1 also tests knowledge of order of operations. Examples 3 and 4 are problem solving items. Example 3 requires an understanding of exponential growth while Example 4 requires students to discern a pattern and choose the best answer. Each incorrect response in each item is based upon the *mathematical errors* that students might make while working problems like these. A description of the errors associated with Example 2 shows what this means.

A. The student reads the problem as $3n = 63$ and solves that problem by dividing 3 by 63.
B. This is the correct answer.
C. The student reads the problem as $3n = 63$ and solves that problem correctly.
D. The student reads the problem as $3n = 63$ and solves that problem by taking the product of 3 and 63.
E. The student solves the problem $n^{1/3} = 63$ correctly.

Figure 10. Description of the Errors Students Might Make While Solving $n \times n \times n = 63$

Development of test questions when calculator or computer use is expected requires considerable skill and a thorough knowledge of the technological tools that students will use. If the suggestions I have made about the ways in which we should use the technologies we choose while teaching are followed, then we will be able to develop reliable, valid tests for our students.

Conclusion

This is an exciting–and disturbing–time for collegiate mathematics. We have an opportunity to restructure our curriculum, to develop new pedagogies, and to test students more accurately if we effectively and appropriately apply present and emerging technologies. On the other hand because of their nature technology tools will probably cause us to reexamine carefully the ways we teach and as a result, to abandon some of those ways. Before the beginning of the 21st century I hope that college mathematics classrooms will be more exciting places in which students are learning mathematics better than ever before–including what each of us remembers as "the good old days" when we were initially learning mathematics!

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