Using Computer Graphing to Enhance the Teaching and Learning of Calculus and Precalculus Mathematics

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Today's technology is dramatically changing the way mathematics is valued and used in the "real world". Corresponding change that recognizes how technology can be used to enhance the teaching and learning of mathematics is needed. The technology based approach to the teaching and learning of mathematics described in this paper was piloted for two years and field tested for one year in The Ohio State University Calculator and Computer Precalculus (C^2PC) Project [9]. The C^2PC teachers are using two important technology driven instructional models. Students participate in an interactive lecture-demonstration instructional model in a classroom containing a single computer. Computer laboratories and classrooms where students have graphing calculators provide a setting for a guided-discovery instructional model. Teachers use a carefully prepared sequence of questions and activities to help students understand or discover important mathematical concepts.

The C^2PC project was supported in pilot by the Ohio Board of Regents and British Petroleum and was supported in field test by the NSF [1]. Besides the authors, Alan Osborne and Gregory Foley from the College of Education are part of the C^2PC project team. The C^2PC approach and textbook, *College Algebra and Trigonometry, A Graphing Approach* [13] will be used in all college algebra and trigonometry courses at Ohio State beginning Autumn Quarter, 1989. Ohio State has been on the leading edge of using technology in freshman mathematics instruction for over 15 years [19].

Computer Based Graphing

The standard traditional approach uses arithmetic and algebraic information to produce graphs of functions and relations and to develop geometric intuition important in the study of calculus and advanced mathematics. The C^2PC approach uses computers or graphing calculators (really pocket computers) to quickly obtain accurate graphs to provide many more examples and further strengthen geometric understanding and foreshadow the study of calculus.

The graphing technology is under student control. Students can choose the viewing window or rectangle in which to display a graph. The *viewing rectangle* $[L, R]$ by $[B, T]$ is the rectangular portion of the coordinate plane determined by $L \leq x \leq R$ and $B \leq y \leq T$ (Figure 1). The $[-10, 10]$ by $[-10, 10]$ viewing rectangle is called the *standard viewing rectangle*.

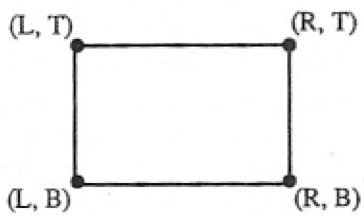

Figure 1. The Viewing Rectangle $[L, R]$ by $[B, T]$

Graphing calculators and the graphing software *Master Grapher* [21] used by C^2PC students has important zoom-in and zoom-out features. *Master Grapher* contains powerful function, conic, polar, parametric, and two variable surface graphing utilities. Versions are available for the IBM, Apple IIe, c or GS, and the Macintosh computer. The graphs in this article were created using the Macintosh version of *Master Grapher*.

Zoom-in is a process of framing a small rectangular area within a given viewing rectangle, making it the new viewing rectangle, and then quickly replotting the graph in this new viewing rectangle. This feature permits the user to create a sequence of nested rectangles that "squeeze down" on a key point of a graph. Zoom-in is very useful for solving equations, systems of equations, inequalities, and for determining maximum and minimum values of functions. The graphing zoom-in process yields answers as accurate as any numerical method.

Zoom-out is a process of increasing the absolute value of the viewing rectangle parameters.

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The zoom-out process is useful for examining limiting, end behavior of functions and relations and for determining "complete" graphs. A *complete graph* is the entire graph displayed in an appropriate viewing rectangle, for example, $x^2+y^2=16$ in $[-10,10]$ by $[-10,10]$; or a *portion* of a graph displayed in an appropriate viewing rectangle which shows all of the important behavior and features of the graph, for example, $f(x)=x^3-x+15$ in $[-10,10]$ by $[-10,30]$. Of course, it is possible to create a function for which you cannot determine *one* viewing rectangle that gives a complete graph. Thus, several viewing rectangles may be needed to describe a complete graph.

The Role of Graphing in Calculus

Calculus textbook authors assume that students have control of graphing. Graphs of functions are often used to illustrate the definition of limit. For example, the following excerpt taken from a standard calculus textbook appeared right after the definition of limit of a function.

> "The function f defined by $f(x)=\frac{1}{x}$ provides an illustration in which no limit exists as x approaches 0 . If x is assigned values closer and closer to 0 (but $x \neq 0$), $f(x)$ increases without bound numerically as illustrated by Figure 2."

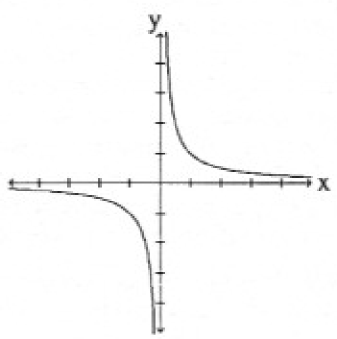


Figure 2. The graph of $f(x)=\frac{1}{x}$

Here the author assumes that students have enough understanding and control of the graph of $f(x)=\frac{1}{x}$ to use it to help understand the concept of limit. In reality, many calculus students are not able to produce a correct sketch of this graph. This important subtle notion of limit is further confounded by lack of understanding about graphs of functions.

The same textbook uses Figure 3 to illustrate the meaning of $\lim_{x\to a} f(x)=L$. Notice the depth of understanding about graphs required by this figure. Many entering calculus students are not even able to correctly produce the graph of a quadratic function.

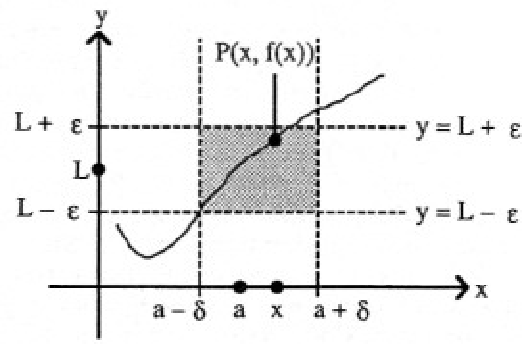
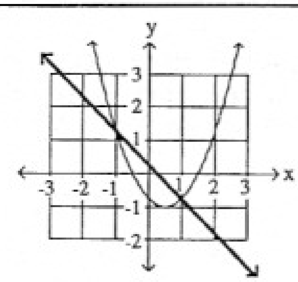


Figure 3. A Geometric Illustration of $\lim_{x\to a} f(x)=L$

The following test item (Figure 4) appeared on the Second International Mathematics Study (SIMS) 12th grade test [18]. United States 12th grade calculus students scored 29% on the pretest and 44% on the posttest on this item. United States precalculus students scored 22% on the pretest and 31% on the posttest on this item. The international posttest average score on this item was 58%.



For what values of x does the function represented by the straight line $\overleftrightarrow{MN}$ exceed the quadratic function?

A. $-1 < x < 1$

B. $x < -1$ or $x > 1$

C. $-\frac{3}{4} < x < 1\ \frac{1}{4}$

D. $x > 0$

E. $x > y$

Figure 4. A SIMS Test Item.

All students must acquire a better understanding about graphs of functions in precalculus if more students are to be successful in calculus. It is our position that the proper use of technology in precalculus courses can significantly enhance student understanding and facility with graphing. This in turn will lead to better understanding of important concepts in calculus.

Changes in Mathematics as a Consequence of Technology

The role of algebraic manipulation. Some leaders in mathematics education call for drastic reduction in time spent on algebraic manipulation. We are convinced that the amount of time on this topic should be reduced, but are not ready to completely abandon algebraic manipulation. First hand observations have convinced us that the use of technology helps student gain new understanding about and provides motivation for important algebraic processes. Graphing gives a geometric interpretation to algebraic procedures. We have found that students are willing, even eager, to perform both arithmetic and algebraic procedures when those procedures answer questions generated by graphs.

Example 1. Determine the real zeros, the end behavior, and draw a complete graph of

$$f(x) = \frac{x^3 - 7x^2 - 12x + 54}{x - 1}.$$

Solution. It can be shown that the graph of f in Figure 5 is complete. One important connection students need to make is that the zeros of f are the same as the x-intercepts of the graph of f. Because the graph is complete, we can be sure that there are three real zeros. Zooming in around an x-intercept to find a zero helps establish and solidify this connection. There appears to be a zero near $x = -3$. We can use this geometric observation to motivate students to divide the numerator of f by $x+3$ or to compute $f(-3)$. Thus, arithmetic and algebraic ideas can be motivated by a graph.

If we zoom in around the zero of f between 2 and 3 a few times we obtain the graph in Figure 6 and can read that 2.354 is a reasonable approximation. We say that 2.354 is a zero of f with error at most 0.01, the distance between the horizontal scale marks in Figure 6.

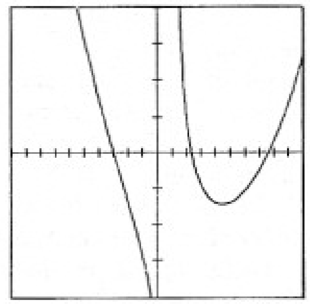


Figure 5. A Complete Graph of $f(x) = \frac{x^3-7x^2-12x+54}{x-1}$ in $[-10, 10]$ by $[-40, 40]$

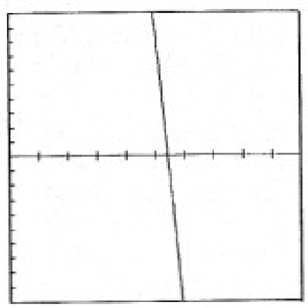


Figure 6. A Zoom-In View of a Zero of $f(x) = \frac{x^3-7x^2-12x+54}{x-1}$ in $[2.3, 2.4]$ by $[-0.1, 0.1]$

In general, the *error* in using a point (x, y) in the viewing rectangle $[L, R]$ by $[B, T]$ to approximate any point (a, b) in the viewing rectangle is *at most* $R - L$ for x and $T - B$ for y. Of course, there are better error bounds possible by overlaying a lattice in a viewing rectangle or by using scale marks appearing in a viewing rectangle. We can use zoom-in to find that the other positive zero of f is 7.645 with error at most 0.01.

The ability to quickly obtain a graph of $y = f(x)$ makes it very natural to discuss the geometric interpretation of solving the equation $f(x) = 0$ or the inequality $f(x) > 0$. Solving equations and inequalities using a zoom-in procedure soon becomes an easy geometric problem of finding x-intercepts, or when one graph is above or below another, or when one graph is above or below the x-axis.

If we zoom out a few times we can obtain the graph in Figure 7. Notice this graph looks very much like the graph of $y = x^2$. In fact, if we overlay the graph of $y = x^2$ the two graphs will appear coincident. This is the geometric meaning of end behavior; the behavior of a function for large $|x|$. The graph of $y = x^2$ is called an *end behavior model* of the rational function f. With selected examples of rational functions as a guide, students can be led to conjecture the end behavior of a rational function. Our students quizzed us for a way to determine, without using zoom-out, the end behavior of such functions. This discussion led to the introduction of the *end behavior asymptote* of a rational function. Their attention was held as we used *long division* to rewrite f as follows:

$$f(x) = x^2 - 6x - 18 + \frac{36}{x-1}$$

Our students were then able to use this form to draw a correct rough sketch of f by replacing f by the end behavior *asymptote* $y = x^2 - 6x - 18$ for values of x away from $x = 1$ and the hyperbola $y = \frac{36}{x-1}$ for x near 1. This algebraic procedure and added insight was due to the ability of students to produce large numbers of graphs in a short period of time. We have found that we can do more with algebraic manipulation when it is *not* the focus of a lesson.

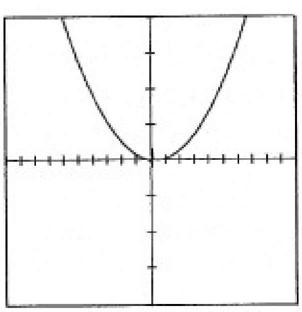


Figure 7. A Zoom-Out View of $f(x) = \frac{x^3-7x^2-12x+54}{x-1}$ in $[-100, 100]$ by $[-4000, 4000]$

Establishing connections among problem situations, algebraic representations, and geometric representations. The easy availability of geometric representations gives students and teachers the opportunity to explore and exploit the connections between algebraic and geometric representations and makes multiple representations of problem situations possible. Analyzing the problem situation through both algebraic and geometric representations deepens student understanding about the problem situation. Instead of the usual negative attitude about word problems, students gain more confidence about problems with the added technique of analyzing and solving them graphically. Word problems seem less mysterious and not as formidable with the addition of a geometric representation and powerful graphic problem solving methods.

Example 2. Squares of side length x are removed from a 8.5 inch by 11 inch piece of cardboard (Figure 8). A box with no top is formed by folding along the dashed lines in Figure 8.

(a) Express the volume V of the box as a function of x.

(b) Draw a complete graph of the algebraic model V.

(c) Which portion of the geometric model (graph) in (b) represents the problem situation.

(d) Determine x so that the box has maximum possible volume and find this maximum volume.

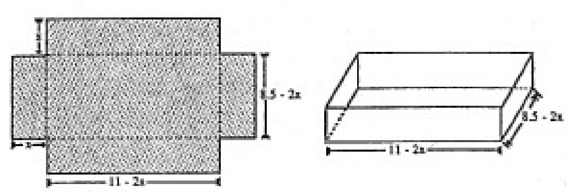


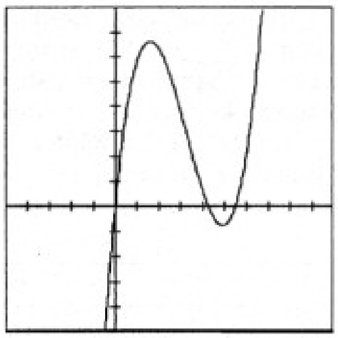
Figure 8. The Box Problem

Solution.

(a) The formula $V = LWH$ can be applied to obtain the volume V as a function of x. The height is x, the length is $11 - 2x$, and the width is $8.5 - 2x$. Thus, $V(x) = x(8.5 -$

$2x)(11-2x)$ is an *algebraic representation* of the volume as a function of x.

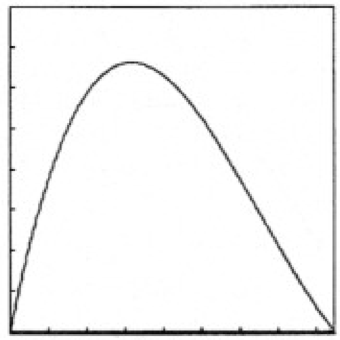
(b) A complete graph of $y = V(x) = x(8.5 - 2x)(11 - 2x)$ is shown in Figure 9. Students will need to experiment with different viewing rectangles until a complete graph is determined. Students would be expected to have had considerable computer based experience graphing cubic polynomials before investigating this problem.



[-5, 10] by [-50 , 80]

Figure 9. A Complete Graph of $V(x) = x(8.5 - 2x)(11 - 2x)$

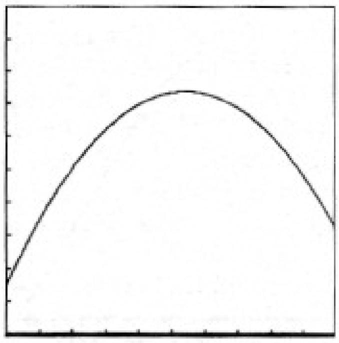
(c) The physical limitations inherent in removing a square of side length x implies that x must be positive. Because the smaller side of the rectangular piece of cardboard is 8.5, $2x$ must be less than 8.5, or x must be less than 4.25. Thus, the values of x that make sense in this problem situation are $0 < x < 4.25$. This means that only the portion of the graph in Figure 9 in the first quadrant that is above the x-axis with $x < 4.25$ represents the problem situation. Therefore, the graph in Figure 10 is a *complete graph of the problem situation*.



[-0, 4.25] by [0 , 80]

Figure 10. A Complete Graph of the Box Problem

(d) Figure 9 strongly suggests that there is a maximum value of V of about 66 and it occurs when x is about 1.6. We find that considerable discussion is necessary for students to readily associate the coordinates of the "maximum point" with a solution to this real world "maximization" problem. First, the connection between the coordinate representation of points (a, b) of the graph of V and $b = V(a)$ must be established. That is, in (a, b), a represents a possible side length of a removed square and b the corresponding volume of the resulting box only for certain values of a and b. Such discussion helps establish the connections among the graphical representation, the algebraic representation $y = V(x)$, and the problem situation representation. Now, if (a, b) are the coordinates of the highest point, students can see that the maximum volume is $b = V(a)$ and that a is the side length of the removed square. Such connections must be carefully developed with many examples during the school year. Once this kind of activity is well established, it is easy to move to zoom-in as a procedure for determining very accurate solutions to these types of problems. Figure 11 illustrates the last viewing rectangle used in a zoom-in process. The figure shows that the volume is 66.14823 with error at most 0.0001 and the associated value of the side length of the removed square is 1.5854 with error at most 0.001.



[1.58, 1.59] by [66.1475, 66.1485]

Figure 11. A Zoom-In View of the Relative Maximum of V

Example 2 illustrates how graphing can be used by precalculus students to foreshadow the study of calculus. Furthermore, graphing technology removes the barriers imposed by limited algebraic techniques available to precalculus and calculus students.

Problems need no longer be contrived. Realistic problems are accessible to students much earlier in their study of mathematics through the use of technology. Lack of familiarity or facility with algebraic techniques need no longer be a barrier to quality problem solving activity by students.

Example 3. A couple can afford to pay $600 per month for a 25 year home loan. What APR (annual percentage rate) interest rate will permit them to purchase a $65,000 home?

Solution. Let x be the *monthly* interest rate. Then $12x$ is the APR rate. It is not difficult to establish that x is given by [20]

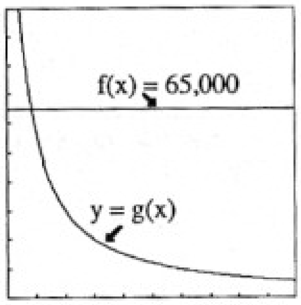
$$65,000 = 600\frac{1-(1+x)^{-300}}{x}.$$

Because there is no closed form solution to this equation, a numerical method is required to find a solution. In fact, a graphing based method is quite natural. One way to solve $f(x) = g(x)$ graphically is to simply graph $y = g(x)$ and $y = f(x)$ in the same viewing rectangle and then look for points common to both graphs (points of intersection).

Let $f(x)$ be the left-hand side and $g(x)$ the right-hand side of the above equation. In this problem it is particularly important to choose a reasonable first viewing rectangle. The problem situation indicates that we need only graph f and g in the first quadrant. (Why?) It must be established that the y values represents possible dollar amounts for the loan. Thus, the maximum y value for a viewing rectangle must be greater than 65,000. Because x is a monthly interest rate, it is reasonable to assume x is less than 0.1 (10% per month). Figure 12 shows complete graphs of

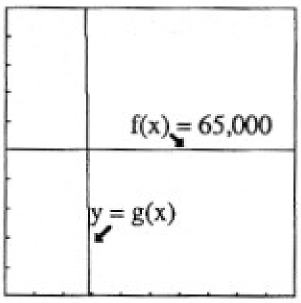
$$f(x) = 65,000 \quad \text{and} \quad g(x) = 600\frac{1-(1+x)^{-300}}{x}$$

in the $[0, 0.1]$ by $[0, 100,000]$ viewing rectangle. That there is only one solution is readily apparent. The graph in Figure 12 suggest x is about 0.01. Zoom-in can be used to determine that the monthly interest rate x is 0.008503 with error less than 0.00001 as shown in Figure 13. Thus, the desired APR rate of the home loan is 10.20%.



[0, 0.1] by [0, 100,000]

Figure 12. The graphs of $f(x) = 65,000$ and $g(x) = 600\frac{1-(1+x)^{-300}}{x}$



[0.0085, 0.00851] by [64999.5, 65000.5]

Figure 13. A Zoom-In View of the Loan Problem

Graphing surfaces (functions of two variables) is easily accessible to precalculus and calculus students when technology is used. Obtaining graphs of surfaces by hand is a difficult task for both student and teacher. Students have a good bit of trouble visualizing in three dimensions. Teachers have a hard time producing quick, accurate graphs of functions of two variables. The graphing software *Master Grapher* used in C^2PC has a powerful utility which allows the user to obtain accurate graphs of functions of two variables. The user can obtain the graphs for $a \le x \le b$, $c \le y \le d$, and $e < z < f$, and then choose an arbitrary point in three dimensional space from which to view the graph. Once the first graph is drawn the points are stored in an array so that the graph can be redrawn quickly from different views. The user can choose any point in three dimensional space from which to view the graph. The resolution of a graph is under user control.

This three dimensional grapher allows the user to interactively explore the behavior of surfaces. Local maximum and minimum values of functions of two variables can be investigated graphically. The grapher can help students deepen understanding and intuition about functions of two variables. It can provide a geometric representation of multidimensional problem situations to go along with an algebraic representation. The connections between these two representations can be also explored and exploited to gain better understanding about problem situations in a manner similar to using a single variable function grapher.

First the user chooses a region of three dimensional space in which to draw a graph of a function of two variables. The set $\{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ e \le z \le f\}$ is called the *viewing box* $[a,b]$ by $[c,d]$ by $[e,f]$.

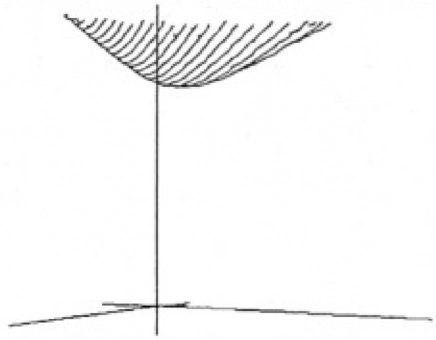
Next, the user decides how to view the graph contained in the selected viewing box. Two points can be selected. The point at which the user places his/her "eye" is called the *viewing point*. The point at which the view of the eye is directed is called the *aiming point*.

Example 4. A box with no lid has volume 6 ft^3. Determine the dimensions of a box with minimum surface area.

Solution. Let x be the width of the box and y the length. The height h of the box is given by $h = \frac{6}{xy}$. If z is the surface area of the box, then

$$z = 2hx + 2hy + xy \quad \text{or} \quad z = \frac{12}{x} + \frac{12}{y} + xy.$$

Because x, y, and z must be positive, we need to investigate the graph only in the first octant. Figure 14 gives the graph of z in the viewing box $[0,10]$ by $[0,10]$ by $[0,20]$ with aiming point $(5,5,10)$ and viewing point with spherical coordinates with respect to the aiming point of ρ (distance) $= 50$, θ (rotation) $= 30°$, and ϕ (elevation) $= 90°$.



Aiming point (5, 5, 10); Distance 50; Elevation 90°; Rotation 30°; Viewing Cube [0, 10] by [0, 10] by [0, 20]

Figure 14. The graph of $z = \frac{12}{x} + \frac{12}{y} + xy$

Notice the graph in Figure 14 suggests the existence of a relative minimum. *Master Grapher* allows the user to interrupt the graphing process and read the coordinates of a point. We can use this technique to estimate the coordinates of the lowest point in Figure 14 to be (2.3, 2.3, 15.7). The actual answer can be shown to be $(\sqrt[3]{12}, \sqrt[3]{12}, 3\sqrt[3]{144}) = (2.289\ldots, 2.289\ldots, 15.724\ldots)$.

Motion Simulation. In an article in *The American Mathematical Monthly*, Neal Koblitz [17] discussed four complicated real-world problems that are not typically solved in calculus textbooks. One problem is especially intriguing to us because it can be simulated and studied with a parametric equation graphing utility. Furthermore, an elementary, non–calculus geometric solution can be obtained with a function graphing utility. The solution suggested by Koblitz involves determining a *derivative to minimize* an expression and then involves *solving a complicated equation iteratively using Newton's method.*

Example 5. You are standing on the ground at point B (Figure 15), a distance of 75 feet from the bottom of a ferris wheel 20 feet in radius. Your arm is at the same level as the bottom of the ferris wheel. Your friend is on the ferris wheel, which makes one revolution (counterclockwise) every 12 seconds. At the instant when she is at point A you throw a ball to her at 60 ft/sec at an angle of 60°

above the horizontal. Take $g = -32$ ft/sec^2, and neglect air resistance. Find the closest distance the ball gets to your friend ... accurate to within $\frac{1}{2}$ foot. [17, p. 256]

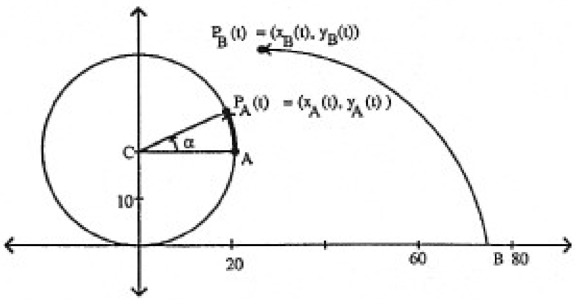


Figure 15. The Ferris Wheel Problem

Solution. The problem situation can be nicely simulated using the parametric equation graphing utility of *Master Grapher* or by using a graphing calculator with a short program to graph parametric equations [7]. The minimization problem can also be easily solved by a non-calculus, graphing zoom-in procedure. The ferris wheel is placed in a rectangular coordinate system with a diameter along the y-axis, the bottom at the origin, and the top at the point $(0, 40)$ (Figure 15). The ferris wheel is a circle with center $C(0, 20)$. Let t be the time in seconds the ball is in flight, $P_A(t) = (x_A(t), y_A(t))$ the position of the friend on the ferris wheel at time t, and $P_B(t) = (x_B(t), y_B(t))$ the position of the ball at time t. Notice that $P_A(0) = A = (20, 20)$ and $P_B(0) = B = (75, 0)$. It is easy to show, using only right triangle trigonometry and high school physics, that $P_A(t)$ and $P_B(t)$ are given by

$$x_A(t) = 20\cos\left(\frac{\pi t}{6}\right)$$

$$y_A(t) = 20 + 20\sin\left(\frac{\pi t}{6}\right)$$

and

$$x_B(t) = 75 - 30t$$

$$y_B(t) = 30\sqrt{3}t - 16t^2.$$

Our parametric graphing utility allows the student to graph any relation $(x(t), y(t))$ defined parametrically by specifying a t interval as [tmin, tmax], and a *viewing rectangle* $[a, b]$ by $[c, d]$. To simulate the problem situation, set tmin $= 0$ and let tmax take on different values and observe the paths of the friend and the ball. The two paths are plotted *simultaneously* producing an excellent simulation of the problem situation. Figure 16 show actual screen dumps from a Macintosh computer of the four simulations given by tmax $1, 1.5, 2$, and 3. The same results can be obtained using *Master Grapher* on an IBM PC or Apple II computer or a graphing calculator [14]. Each simulation takes less than 5 seconds!

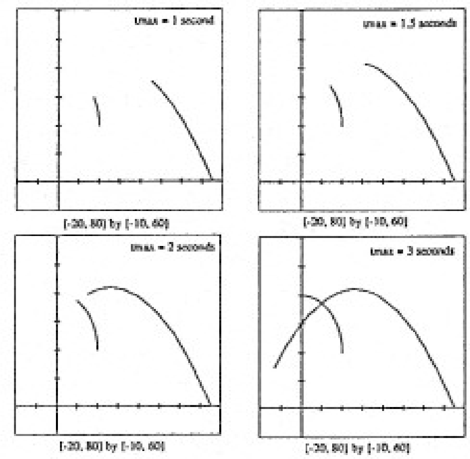


Figure 16. Simulations of the Ferris Wheel Problem for $t$max $1, 1.5, 2$, and 3.

Figure 16 indicates that the two *paths* have a common point. However, the values of t (time) that produces the common point are *different* for each set of parametric equations. It is really the *endpoints* of both curves that are of interest. The solution to the problem can be found by determining a value of t that *minimizes* the distance between the ball and the position of the friend on the ferris wheel. A parametric equation graphing utility that produces simultaneous graphs can be used to approximate the solution using "guess and check." It is easily shown that a value of tmax between 2.1 and 2.3 seems to yield the minimum distance.

The speed of computer graphing makes a "guess and check" simulation method possible and appropriate (some students even say "fun!") for mathematical exploration. The next figure gives a closer view of the solution. Each graph is drawn in the $[5, 15]$ by $[33, 42]$ viewing rectangle. The t range for the graphs are $[0, 2.1]$, $[0, 2.15]$, $[0, 2.2]$, and $[0, 2.3]$, respectively. It is easy to see from these graphs that the common point of the two paths is reached at different times. These static figures fail to do justice to the insight gained by observing the dynamic, "real time" computer generated simulation. That is, by observing simultaneously the position of the ball and the friend on the ferris wheel as t (time) increases.

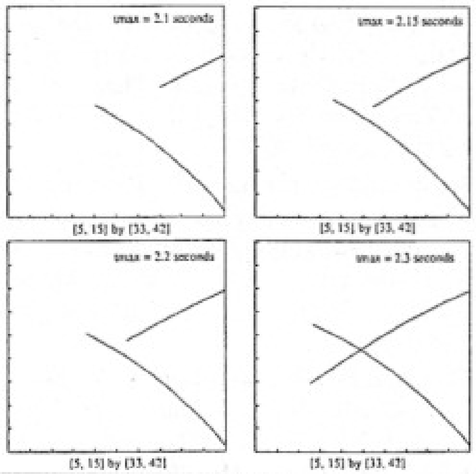


Figure 17. Simulations of the Ferris Wheel Problem for $t\max$ 2.1, 2.15, 2.2, and 2.3

By estimating the distance between the end points of the two paths, a student can quickly determine that the minimum distance occurs when t is near 2.2 seconds, and that the actual minimum distance is probably less that 2 feet. Notice that the scale marks in Figure 17 are one unit in length.

The distance formula can be used to write the distance D between $P_A(t)$ and $P_B(t)$ as a function of time t.

$$
\begin{aligned}
D(t) &= \sqrt{(x_A(t) - x_B(t))^2 + (y_A(t) - y_B(t))^2} \\
&= [(20\cos\left(\frac{\pi t}{6}\right) - 75 + 30t)^2 \\
&\quad + (20 + 20\sin\left(\frac{\pi t}{6}\right) - 30\sqrt{3}\,t + 16t^2)^2]^{\frac{1}{2}}
\end{aligned}
$$

Solving the equation involving the derivative ($D'(t) = 0$) is very difficult (try it "by hand"). However, the minimum value of the function D can be found easily and quickly by drawing a graph of $y = D(t)$ and using a graphing *zoom-in* process to determine the coordinates of the minimum. Figure 18 gives the graph of $y = D(t)$ for $0 \leq t \leq 3$. Figure 19 is the result after several iterations of the graphic zoom-in procedure.

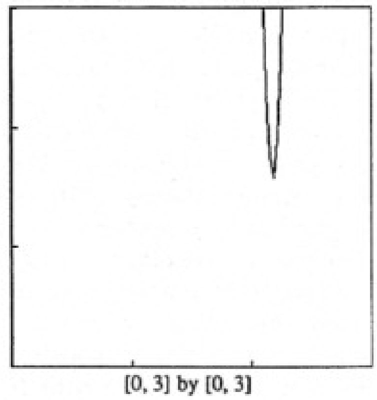


Figure 18. The Graph of $y = D(t)$

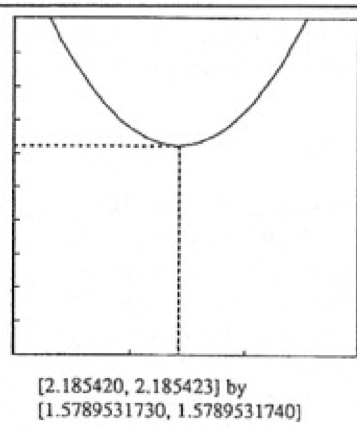


Figure 19. A Zoom-In View of the Graph of $y = D(t)$

The coordinates $(2.1854214, 1.5789531736)$ of the local minimum of $y = D(t)$ can be read from Figure 19. The error in the first coordinate is at most $0.000001 = 10^{-6}$, the distance between horizontal scale marks. The error in the second coordinate is at most 10^{-10}, the distance between vertical scale marks. Thus, the minimum distance is 1.5789531736 feet with error at most 10^{-10} feet, and occurs when t is 2.1854214 feet with error at most 10^{-6} seconds.

Other interesting problems can be posed and solved using this computer simulation approach. For example, how could the angle of elevation be adjusted so that the ball comes within *very* easy catching distance (say 6 inches)? How close together will two balls come if thrown at the same time by two people facing each other (vary the distance between the two people, the angles of elevation, and the initial velocities)?

Students are often exposed for the first time to important topics such as parametric and polar equations and graphing 3 dimensional surfaces in calculus. This makes the task of a calculus teacher even more ominous because students must quickly learn and apply these ideas. Little or no attention is given to these topics in precalculus and calculus because of the difficulties of graphing such curves by hand. Technology permits students to quickly determine a graph and to discover the role of a parameter by experimentation. More realistic and interesting problems are possible because of the speed and power of technology and the fact that algebraic complication is not a factor when technology is used. Important new approaches (such as the computer simulation demonstrated in the previous example) are possible with technology. Students need a rich intuitive background prior to the study of calculus and other advanced mathematics and science courses. Technology deepens the level of student understanding and reduces the time necessary to acquire such understanding.

References

1. Demana, Franklin and Bert K. Waits. "Enhancing Problem Solving Skills in Mathematics Through Microcomputers." *Collegiate Microcomputer* 5 (February 1987), 72–75.
2. Demana, Franklin and Bert K. Waits. "Problem Solving Using Microcomputers." *The College Mathematics Journal* 18 (May 1987) 236–241.
3. Demana, Franklin and Bert K. Waits. "Solving Problems Graphically using Microcomputers." *The UMAP Journal* 8 (Spring 1987) 1–7.
4. Demana, Franklin and Bert K. Waits. "Microcomputer Graphing–A Microscope for the Mathematics Student." *School Science and Mathematics* 88 (March 1988), 218–224.
5. Demana, Franklin and Bert K. Waits. "Pitfalls in Graphical Computation, or Why a Single Graph Isn't Enough." *The College Mathematics Journal* 19 (March 1988) 177.
6. Demana, Franklin and Bert K. Waits. "Manipulative Algebra–The Culprit or the Scapegoat?" *The Mathematics Teacher* 81 (May 1988), 332.
7. Demana, Franklin and Bert K. Waits. "Using Graphing Calculators to Enhance the Teaching and Learning of Precalculus Mathematics: Teacher Inservice Worksheets for the CASIO Graphing Calculator." Department of Mathematics. Columbus (August, 1988).
8. Demana, Franklin and Bert K. Waits. "An Application of Programming and Mathematics: Writing a Computer Graphing Program." *The Journal of Computers in Mathematics and Science Teaching* 7 (Summer 1988), 47–48.
9. Demana, Franklin and Bert K. Waits. "The Ohio State University Calculator and Computer Precalculus Project: The Mathematics of Tomorrow Today!" *The AMATYC Review* 10 (Fall 1988), 46–55.
10. Demana, Franklin and Bert K. Waits. "Using Computer Graphing to Enhance the Teaching and Learning of Calculus and Precalculus Mathematics". MAA Minicourse Monograph (Atlanta, January, 1988); (Phoenix, January, 1989).
11. Demana, Franklin and Bert K. Waits. "Computers and the Rational Root Theorem–Another View." *The Mathematics Teacher* 80 (February, 1989), 124–125.
12. Demana, Franklin and Bert K. Waits. "A Computer Graphing Based Approach to Solving Inequalities." *The Mathematics Teacher*, (May, 1989), 327–331.
13. Demana, Franklin and Bert K. Waits. *College Algebra and Trigonometry, A Graphing Approach*. Addison-Wesley Publishing Company, Reading, MA, 1990.
14. Demana, Franklin, Bert K. Waits, Greg Foley, and Alan Osborne (1990) *Graphing Calculator and Computer Graphing Laboratory Manual* to accompany *College Algebra and Trigonometry, A Graphing Approach* Addison–Wesley. Reading, MA, 1990.
15. Demana, Franklin and Bert K. Waits. "A Hidden Case of Negative Amortization." In press (*The College Mathematics Journal*).
16. Demana, Franklin and Bert K. Waits. "Around the Sun in a Graphing Calculator." In press (*The Mathematics Teacher*).
17. Koblitz, Neal. "Problems That Teach the Obvious but Difficult," *The American Mathematical Monthly*, Volume 95, Number 3 (March 1988) 254–257.
18. Travers, Kenneth J., Director. *Second International Mathematics Study, Detailed National Report United States.* Champaign, Illinois: Stipes Publishing Company, 1985.
19. Waits, Bert K. and Joan Leitzel. "Hand Held Calculators in the Freshman Mathematic Classroom". *American Mathematical Monthly* 83. November, 1976.

19. Waits, Bert K. and Joan Leitzel. "Hand Held Calculators in the Freshman Mathematic Classroom". *American Mathematical Monthly* 83. November, 1976.

20. Waits, Bert K. "The Mathematics of Finance Revisited Through the Hand Calculator." In *Applications of Mathematics, 1979 Yearbook of the National Council of Mathematics*, The National Council of Teachers of Mathematics, Washington, D.C., 1987.

21. Waits, Bert K., and Franklin Demana. *Master Grapher* (Computer Software: IBM, Apple II, and Macintosh Versions). Addison-Wesley Publishing Company, Reading, MA, 1990.