3.2 Cofactor Expansion

DEF ($\rightarrow$ p. 152) Let $A = [a_{ij}]$ be an $n \times n$ matrix.

- M_{ij} denotes the $(n-1) \times (n-1)$ matrix of A obtained by deleting its i-th row and j-th column.
- $\det(M_{ij})$ is called the **minor** of a_{ij}.
- $A_{ij} = (-1)^{i+j} \det(M_{ij})$ is called the **cofactor** of a_{ij}.

EXAMPLE 1 For $A = \begin{pmatrix} 1 & 1 & 4 \\ 0 & -1 & 2 \\ 2 & 3 & 0 \end{pmatrix}$ we have:

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = (-1)(0-4) = 4$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} = (1)(2+4) = 6$$

TH 3.9 ($\rightarrow$ p. 153) Let $A = [a_{ij}]$ be an $n \times n$ matrix. For each $i = 1, \ldots, n$,

- $\det(A) = a_{i1}A_{i1} + \cdots + a_{in}A_{in}$
  (expansion of $\det(A)$ along the i-th row)
- $\det(A) = a_{1i}A_{1i} + \cdots + a_{ni}A_{ni}$
  (expansion of $\det(A)$ along the i-th column)

EXAMPLE 2 In Example 2 ($\rightarrow$ p. 154), the determinant of $A = \begin{pmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{pmatrix}$ was found by

- expansion along the third row, and
- expansion along the first column.

We shall illustrate the expansion along the second column:

$$
\begin{aligned}
\det(A) &= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} + a_{42}A_{42} \\
&= 2(-1)^3 \begin{vmatrix} -4 & 1 & 3 \\ 3 & 0 & -3 \\ 2 & -2 & 3 \end{vmatrix} \\
&\quad + 2(-1)^4 \begin{vmatrix} 1 & -3 & 4 \\ 3 & 0 & -3 \\ 2 & -2 & 3 \end{vmatrix} + 0 + 0 \\
&= -2(0 - 6 - 18 - 0 + 24 - 9) \\
&\quad + 2(0 + 18 - 24 - 0 - 6 + 27) \\
&= -2(-9) + 2(15) \\
&= 48
\end{aligned}
$$

$\boxed{\texttt{TH 3.10}}$ ($\rightarrow$ p. 155) Let $A = [a_{ij}]$ be an $n \times n$ matrix. For each $i \neq k$,

- $a_{i1}A_{k1} + \cdots + a_{in}A_{kn} = 0$
- $a_{1i}A_{1k} + \cdots + a_{ni}A_{nk} = 0$

Outline of the proof:

- Let B be the matrix obtained from A by replacing the kth row with the ith row.
- Expand $\det(B)$ along its kth row. Since $B_{kj} = A_{kj}$ and $b_{kj} = a_{ij}$, this expansion is identical to the LHS of the first formula.
- By Th. 3.3, $\det(B) = 0$. This proves the first formula (the proof of the 2nd formula is identical).

$\boxed{\textbf{DEF}}$ ($\rightarrow$ p. 156) Let $A = [a_{ij}]$ be an $n \times n$ matrix. The adjoint of A is the $n \times n$ matrix

$$\operatorname{adj} A = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

$\boxed{\textbf{TH 3.11}}$ ($\rightarrow$ p. 157) Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$A(\operatorname{adj} A) = (\operatorname{adj} A)A = \det(A)I_n$$

Outline of the proof of $A(\operatorname{adj} A) = \det(A)I_n$:
The (i,j)-element of $A(\operatorname{adj} A)$ is

$$row_i(A) \bullet col_j(\operatorname{adj}A) = a_{i1}A_{j1} + \cdots a_{in}A_{jn}$$

$$= \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\boxed{\textbf{COROLLARY 3.3}}$ ($\rightarrow$ p. 158) If $\det(A) \neq 0$ then

$$A^{-1} = \frac{1}{\det(A)}(\mathrm{adj}A)$$

Equivalent conditions ($\rightarrow$ p.160)

For any $n \times n$ matrix A, the following conditions are equivalent:

1. A is nonsingular.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. A is row equivalent to I_n.
4. For every $n \times 1$ matrix $\vec{b}$, the system $A\vec{x} = \vec{b}$ has a unique solution.
5. $\det(A) \neq 0$.