10.2 The Kernel and Range

DEF ($\rightarrow$ p. 441, 443)

Let $L : V \rightarrow W$ be a linear transformation. Then

(**a**) the **kernel** of L is the subset of V comprised of all vectors whose image is the zero vector:

$$\ker L = \{ \overrightarrow{v} \mid L(\overrightarrow{v}) = \overrightarrow{0} \}$$

(**b**) the **range** of L is the subset of W comprised of all images of vectors in V:

$$\operatorname{range} L = \{ \overrightarrow{w} \mid L(\overrightarrow{v}) = \overrightarrow{w} \}$$

DEF ($\rightarrow$ p. 440, 443)

Let $L : V \rightarrow W$ be a linear transformation. Then

(**a**) L is **one-to-one** if $\overrightarrow{v_1} \neq \overrightarrow{v_2} \Rightarrow L(\overrightarrow{v_1}) \neq L(\overrightarrow{v_2})$

(**b**) L is **onto** W if $\operatorname{range} L = W$.

EXAMPLE 1

Let $L : R^3 \to R^3$ be defined by $L(x, y, z) = (x, y, 0)$. (Projection onto the xy-plane.)

- $\ker L = \{(x, y, z) | (x, y, 0) = (0, 0, 0)\}$
  $\ker L$ consists of (x, y, z) that are solutions of the system
  $$\begin{aligned} x \quad\quad &= 0 \\ y &= 0 \end{aligned}$$
  z is arbitrary, and $x = y = 0$.
  $\ker L = \text{span}\,\{(0, 0, 1)\}$.

- range $L = \text{span}\,\{(1, 0, 0), (0, 1, 0)\}$.

- L is not one-to-one (e.g., $L(1, 2, 3) = L(1, 2, 5) = (1, 2, 0)$.)

- L is not onto (range $L \neq R^3$).

TH (→ Th. 10.4 p. 442, Th. 10.6 p. 443)

Let $L : V \to W$ be a linear transformation. Then

- $\ker L$ is a subspace of V and
- range L is a subspace of W.

TH 10.5 → p. 443

A linear transformation L is one-to-one if and only if $\ker L = \{\vec{0}\}$.

EXAMPLE 2

Let $L : R^2 \to R^3$ be defined by

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix}.$$

- $\ker L =$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Solve the system of equations:

$$\begin{aligned} x_1 \qquad\quad &= 0 \\ x_1 + x_2 &= 0 \\ x_1 + 2x_2 &= 0 \end{aligned}$$

Coefficient matrix:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ has r.r.e.f. } \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\ker L = \{(0,0)\}$. By Theorem 10.5, L is one-to-one.

- range $L = \left\{ \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix} \middle| \text{ for all } x_1, x_2 \right\}$

$$= \left\{ x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \middle| \text{ for all } x_1, x_2 \right\}$$

$$= \text{span}\,\{\underbrace{(1,1,1),(0,1,2)}_{\text{basis for range } L}\} \neq R^3$$

$\Rightarrow L$ is not onto

EXAMPLE 3 Let $L : R^3 \to R^2$ be defined by

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}.$$

- $\ker L =$

  $$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

  The homogeneous system coefficient matrix:

  $$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ has r.r.e.f. } \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

  x_3 is arbitrary, $x_1 = x_3, x_2 = -x_3$.

  $$\ker L = \text{span}\ \{ \underbrace{(1, -1, 1)}_{\text{basis for } \ker L} \}$$

  $$\ker L \neq \{(0,0,0)\} \overset{\text{Th. 10.5}}{\Longrightarrow} L \text{ is not one-to-one.}$$

- range $L = \{\begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix} | \text{for all } x_1, x_2, x_3\}$

$= \{x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} | \text{for all } x_1, x_2, x_3\}$

Find a basis for range $L =$ span $\{(1,0),(1,1),(0,1)\}$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ has r.r.e.f. } \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$\Rightarrow$ range $L =$ span $\{\underbrace{(1,0),(1,1)}_{\text{basis for range } L}\}$.

range $L = R^2 \Rightarrow L$ is onto.

Note that in **EXAMPLE 3** we used r.r.e.f. $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ of the homogeneous system coefficient matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ to determine both the kernel and the range of L. In this case, we had:

- $\ker L$ = null space of A
- range L = column space of A

Recall Th. 6.12 p. 288:
If A is an $m \times n$ matrix then

$$\text{rank}\, A + \text{nullity}\, A = n.$$

TH 10.7 $\rightarrow$ p. 446

Let $L : V \rightarrow W$ be a linear transformation. Then

$$\dim(\ker L) + \dim(\text{range}\, L) = \dim V$$

EXAMPLE 4 ($\rightarrow$ **EXAMPLE 1** from the previous lecture)

$L : P_2 \rightarrow P_3$ is defined by

$L(at^2 + bt + c) = ct^3 + (a+b)t.$

- $\ker L = \{at^2 + bt + c \,|\, ct^3 + (a+b)t = 0\}$

  Set up the homogeneous equation:

$$\begin{array}{rcl} c & = & 0 \\ 0 & = & 0 \\ a + b & = & 0 \\ 0 & = & 0 \end{array}$$

The coefficient matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{has r.r.e.f.} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

b is arbitrary; $c = 0$; $a = -b$.

$\ker L = \text{span}\,\{-t^2 + t\}$.

$\ker L \neq \{0\} \Rightarrow L$ is not one-to-one.

- range $L = \{ct^3 + (a+b)t \mid \text{for all } a,b,c\}$
  $= \{a(t) + b(t) + c(t^3) \mid \text{for all } a,b,c\}$
  $= \text{span}\{\underbrace{t, t^3}_{\text{basis for range } L}\}$

  range $L \neq P_3 \Rightarrow L$ is not onto.

Verify Th. 10.7 for the four examples:

EX	$L : V \to W$	$\dim(\ker L)$	$\dim(\text{range} L)$	$\dim V$
1	$L : R^3 \to R^3$	1	2	3
2	$L : R^2 \to R^3$	0	2	2
3	$L : R^3 \to R^2$	1	2	3
4	$L : P_2 \to P_3$	1	2	3

- $\dim(\ker L)$ = nullity of L
- $\dim(\text{range } L)$ = rank of L.

COROLLARY 10.2 $\to$ p. 443

Let $L : V \to W$ be a linear transformation and $\dim V = \dim W$. L is one-to-one if and only if L is onto.