1   Limits

The notion of a limit is fundamental to the study of calculus. The formal definition of a limit is given on page 66 of LHE. This definition is useful for proving that a given number is (or isn't) the limit of a function. However, finding the limit can often be accomplished by using algebra, by graphing, and/or by evaluating the function. In this laboratory activity, we emphasize the graphical and numerical approaches to finding the limit of a function.

1.1   Activity: Limits

Prerequisite: Read Chapter 1 LHE.

The objective of this activity is to enhance your visual and numerical understanding of the limit concept.

Instructions

After reading the comments and studying the worked examples, open a blank Mathcad document, and create your report there. Answer all the problems identified by your professor. Remember to enter your team's name at the top of the document. Upon completion of the assignment, enter the names of all team members who actively participated in the assignment. Save your work frequently.

Comments

1. This is likely to be the first assignment in which you create your report "from scratch". When Mathcad is initially invoked, it opens an empty document, ready for you to use. After you type a few lines of the document (which, by the way, should include the title of the assignment and the team name), select **Save As...** from the **File** menu to specify the file name and location in your team's home directory (see page 7 for more details).

2. This activity will introduce the built-in Mathcad "if" function, which will enable us to define multipart functions using different expressions depending on a certain condition. The "if" function must have three arguments:

   if(CONDITION, THEN-VALUE, ELSE-VALUE).

   When the "if" function is evaluated, the CONDITION is first checked. If it is true, then the THEN-VALUE is reported as the result, otherwise, the ELSE-VALUE is returned.
   Examples. To define the function

$$f(x) = \begin{cases} \cos x & \text{if } x > \pi \\ \sin x & \text{if } x \le \pi \end{cases}$$

in Mathcad, we use the "if" construct:

$$f(x):=if(x > \pi,\cos(x),\sin(x))$$

Similarly, we could define a function AbsVal(x) which will calculate the value of $|x|$ by definition:

$$AbsVal(x):=if(x \geq 0,x,-x)$$

(Note that the $\geq$ and $\leq$ signs can be found on the second palette.)

3. In Example 1 (and in some Problems) you are asked to play the "**find a rectangle**" game. The game corresponds to a geometric interpretation of the definition of a limit. The rules of this game are as follows:

 – You are <u>given</u> a function, $f(x)$, and three numbers: c, L, and ε (epsilon will always be > 0).

 – You are asked to <u>find</u> (if possible) a positive number δ such that when you draw a rectangle with:

 > center at (c, L)
 > length (in the y direction) $= 2\varepsilon$
 > width (in the x direction) $= 2\delta$

 your $f(x)$ is trapped between the top and the bottom of the rectangle. That is, the graph of $f(x)$ must leave through the sides of the rectangle and not through the top or bottom.
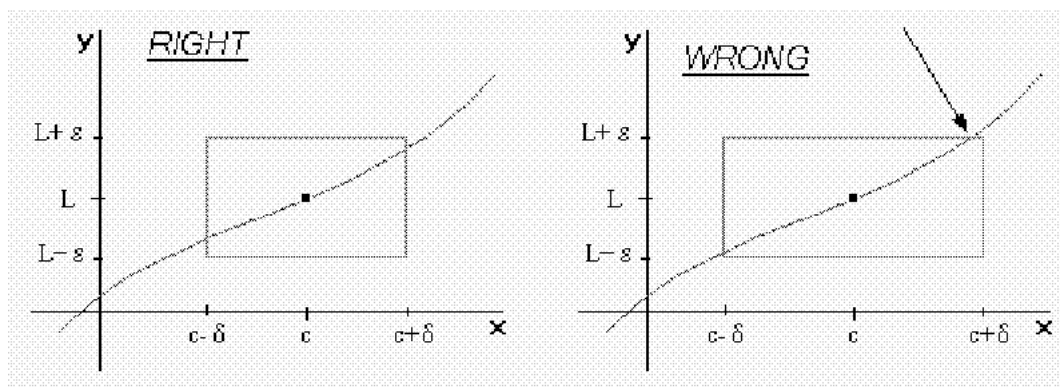
Note: In this game, $x = c$ is the only point where $f(x)$ is allowed be undefined or have a value such that $(c, f(c))$ falls outside the rectangle. This corresponds to the restriction that $|x - c| > 0$ in the definition of the limit (see page 66 LHE).

Example 1

(a) Graph $f(x) = \frac{x\sin(\pi x)}{\sqrt{1-\cos(\pi x)}}$ on the interval $[-4.95, 4.95]$ with step 0.04.

(b) Play the "find a rectangle" game described above with $c = 0$, $L = 0$, $\varepsilon = 2.5$.

(c) Repeat part (b) for the same $f(x), c$ and L, but now with $\varepsilon = 0.7$.

(d) Would you have been able to keep finding values of δ, if we kept supplying you with smaller and smaller positive values of ε?

(e) Play the rectangle game for the same $f(x)$ with $c = 2$, $L = 3$ and $\varepsilon = 6$.

(f) Repeat part (e) using $\varepsilon = 3$.

(g) Use the results of parts (b) through (f), together with the definition of limit (page 66 LHE), to decide whether each statement below is true or false:

  (i)   The limit of $f(x)$ as x approaches 0 is 0.

  (ii) The limit of $f(x)$ as x approaches 2 is 3.

(h) Find <u>all</u> values of x for which $f(x)$ is discontinuous. What kinds of discontinuities are they?

Solution

(a)
$$f(x) := \frac{x \cdot \sin(\pi \cdot x)}{\sqrt{1 - \cos(\pi \cdot x)}} \qquad x := -4.95, -4.91 .. 4.95$$
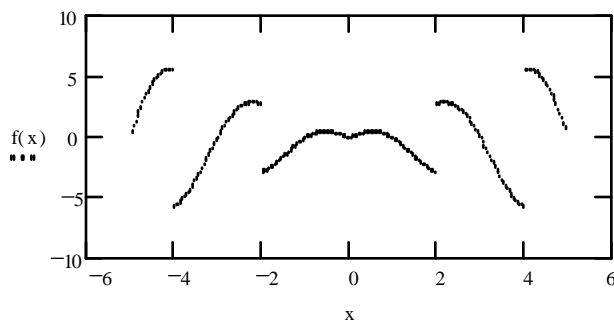
Note that we have set the trace type to points to prevent Mathcad from improperly closing the "gaps" at discontinuities.

(b) First, we get Mathcad to draw two horizontal lines: y=2.5 and y= -2.5, portions of which will become the top and the bottom of our rectangle. (To do it we just set the left and right marker values to Lϵ and L+ϵ, respectively, making sure the "Show markers" option is *on*).

$L := 0 \quad \epsilon := 2.5 \quad c := 0$
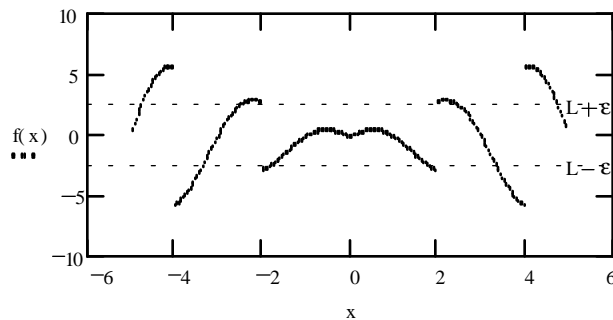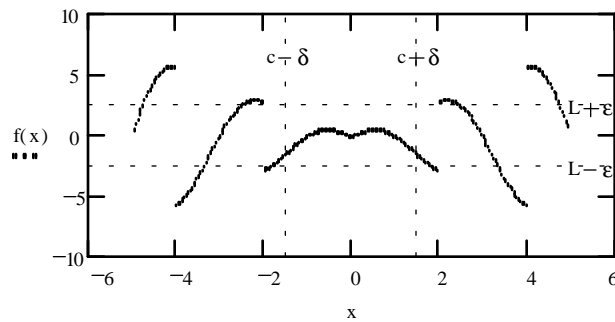
Looks like δ=1.5 will do the job. Let's see:

$\delta := 1.5$

This time we change the values of the left and right markers to c-δ and c+δ, respectively.
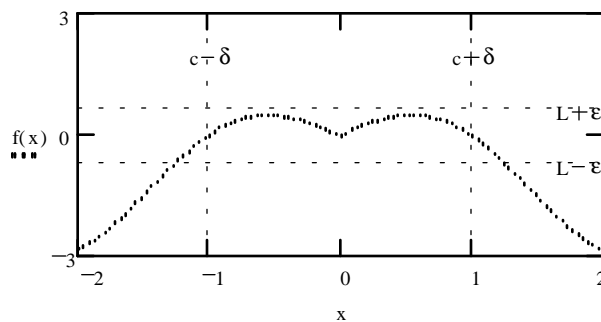Clearly, our rectangle satisfies all the requirements, so we can accept
$\qquad \delta$=1.5
as the answer.

(c) $\epsilon := 0.7 \quad \delta := 1$

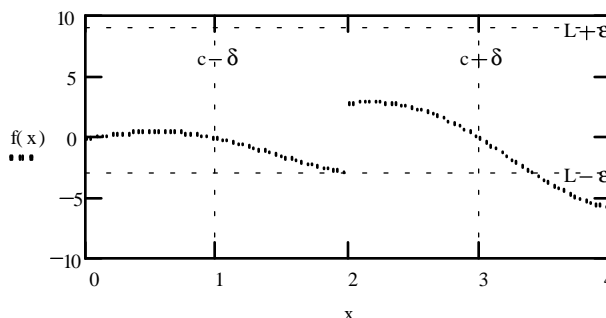(note that we zoomed in on the relevant portion of the graph)

(d) Yes, by choosing a small enough delta for each epsilon, no matter how small, we can get the function "trapped" in the rectangle.

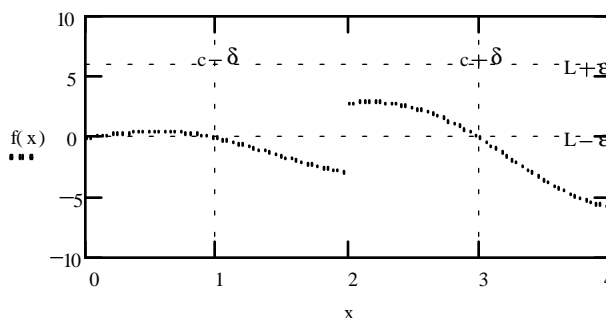**(e)** $c := 2$ $L := 3$ $\varepsilon := 6$ $\delta := 1$

See the graph:

(again, we zoomed in on
the relevant portion of the graph)

(f) $\varepsilon := 3$

No δ will do the job - no matter
how small it is, we will not "trap"
the function graph within the
rectangle.

(g) First statement is true, because of (d); second statement is false, because of (f).

(h) f(x) is discontinuous at x=2n for all integers n.
For n=0 it is a hole discontinuity, otherwise it is a jump discontinuity.

Example 2

(a)   Graph $h(x) = x\sin(1/x)$ on $[-5,5]$ in steps of 0.003 .

(b)   Graphically, what does the limit of $h(x)$ as x approaches infinity appear to be?
Evaluate $h(x)$ for some large x values, to confirm your answer.

(c)   Based on the graph, what does the limit of $h(x)$ as x approaches 0 appear to be?
(Consider zooming in to get a better picture.)
Evaluate $h(x)$ at a few values of x close to 0 to confirm your answer.

(d)   Use Mathcad to evaluate $h(0)$. If Mathcad's answer were correct, would $h(x)$ be continuous at $x = 0$?
Evaluate $h(0)$ by hand. Was Mathcad's answer correct? Is $h(x)$ actually continuous at $x = 0$?
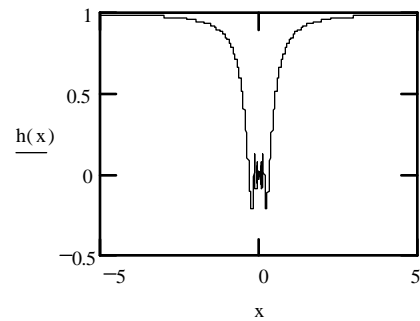
Solution

(a) $h(x) := x \cdot \sin\left(\dfrac{1}{x}\right)$ $x := -5, -4.997.. 5$

(b)   The limit appears to be 1.

$h(10) = 0.998334$

$h(100) = 0.999983$

$h(500) = 0.999999$

(c)   The limit appears to be 0

$h(1) = 0.841471$

$h(0.1) = {}^{-}0.054402$

$h(0.01) = {}^{-}0.005064$

$h(0.001) = 0.000827$

(d) $h(0) = 0$ If this were correct, h(x) would have been continuous at x=0.

However, (0)sin(1/0) cannot be evaluated so that h(0) is undefined. Therefore h(x) is not continuous at x=0.

Problems

In all the problems of this section, the graphs must have "Trace type" switched to "points".

1.   *All the graphs in this problem must have "Show markers" turned ON for both axes.*

(a)   Graph $g(x) = \begin{cases} 2 - x^2 & \text{if } x < 1 \\ 3 - x & \text{if } x \geq 1 \end{cases}$ on $[-3, 10]$ in steps of 0.05 .

(First, define $g(x)$ using Mathcad's "if" function - see Comment 2)

(b)   Play the "Find a Rectangle" game (see Example 1) and try to find a value of δ given:

(i) $c = -1, \ L = 1, \ \varepsilon = 2$

(ii) $c = -1, \ L = 1, \ \varepsilon = 0.5$

(iii) $c = 1, \ L = 2, \ \varepsilon = 2$

(iv) $c = 1, \ L = 2, \ \varepsilon = 0.5$

(Hint: you may want to zoom in on areas of interest.)

(c)   What can you conclude about the limits of $g(x)$ as x approaches -1 and as x approaches 1?

2. (a) Graph $f(x) = (1+x)^{1/x}$ on $[0.01, 20]$ in steps of 0.01 .

    (b) Evaluate $f(x)$ at 0.000001 and 100000 to eight decimal digits.

    (c) Based on the graph obtained in part (a), as well as the values from part (b), estimate the limit of $f(x)$ as x approaches 0 from the right. What about the limit as x approaches infinity?

         Note: The limit of $f(x)$ as x approaches 0 is a very important irrational number which is denoted by the letter e. Mathcad has this number built-in. Try evaluating it (type $e=$) to eight decimal digits, and observe that it is close to the estimate you obtained above.

3. (a) Assign the value 1 to the variable k and graph $f(x) = \frac{k - |x - k|}{x}$ on $[-10, 10]$ in steps of 0.01 with the y-axis extending from -2 to 2.

    (b) With the four objects (the three assignment statements and the graph) on the screen, enclose them all within dotted-line boxes by dragging the mouse. Use the copy and paste commands to create an exact copy below the original region. Below the new copy, paste again.

         In the first copy, change the value of k to 0. In the second copy, change k to -1 (the corresponding graphs will be automatically updated).

    (c) Determine all discontinuities of the three functions graphed above.

         What kinds of discontinuities have you found? Which graph has an invisible discontinuity?

    (d) For all three graphs, what do the limits of $f(x)$ as x approaches infinity appear to be? What if x approaches negative infinity? Confirm your answers by evaluating $f(x)$ at several values of x.

4. A chain of grocery stores advertises that they will double any manufacturer's coupons that are valued at 50 cents or less (coupons worth more than 50 cents are redeemed at face value).

    (a) Use Mathcad's "if" function (see Comment 2) to define a function, $f(x)$, which describes the amount paid to a cashier for a \$2 bag of potato chips while redeeming an \$$x$ coupon from the chips manufacturer. (We assume, realistically, $x \le 2$.)

    (b) Graph $f(x)$ on [0,2] . Note that $f(x)$ is defined for all x on [0,2] even though we are not interested in fractions of a penny.

    (c) Where is $f(x)$ discontinuous? What kind of discontinuity is it? Look at the definition of continuity which appears on page 84 LHE. Which of the three conditions listed there fails to be met?

5. Consider the limit: $\lim\limits_{x \to 3} \frac{x^2 - 3x}{x^2 - 9}$.

    (a) Evaluate the limit by hand.

      (b)   Verify your answer using Mathcad.

            (Evaluate the function for values of x approaching 3 from either side; you should also graph the function.)

6.   Repeat Problem 5 for the limit of Exercise 12 p.82 LHE.

7.   Repeat Problem 5 for the limit of Exercise 14 p.82 LHE.

8.   Repeat Problem 5 for the limit of Exercise 28 p.82 LHE.

9.   Repeat Problem 5 for the limit of Exercise 10 p.92 LHE.

10.   Repeat Problem 5 for the limit of Exercise 34 p.101 LHE.

11.   Repeat Problem 5 for the limit of Exercise 38 p.101 LHE.

1.2   Homework Help

- You may apply the procedure of Problem 5 of this Section to
  - Exercises 9-40 p.82 LHE
  - Exercises 7-22 p.92 LHE
  - Exercises 25-42 p.101 LHE
  - Exercises 1-26 p.102 LHE

- You can use Mathcad to graph each function in Exercises 27-48 p.92 LHE, to illustrate your conclusions regarding the continuity of the function. (Make sure you switch the "Trace Type" to "Points" to prevent Mathcad from connecting points on the graph across discontinuities.)
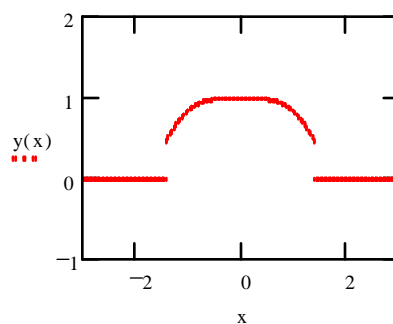
1.3   Questions

1. The function f(x) produced the following output.
Which of the following statements is most likely to be true?

x	f(x)
1.99	5.064
1.991	9.149
1.992	6.16
1.993	9.964
1.994	1.615
1.995	8.733
1.996	9.705
1.997	-3.188
1.998	4.678
1.999	-8.269

x	f(x)
2.01	9.999
2.009	9.999
2.008	9.999
2.007	9.999
2.006	10
2.005	10
2.004	10
2.003	10
2.002	10
2.001	10

A.   The function is not defined at x=2.

B.   The limit of f(x) as x tends to 2 is 10

C.   The limit of f(x) as x approaches 2^- is -9

D.   The limit of f(x) as x tends to 2^+ is 10

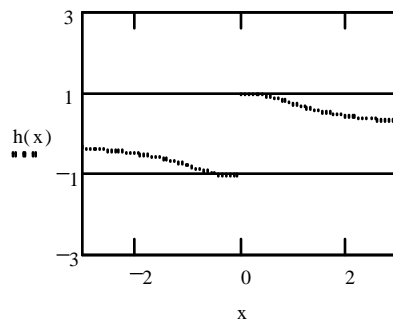E.   The limit of f(x) as x tends to 10 is 2.

2. The function y(x) is plotted below. Determine the limit of y(x) as x tends to 0.

A.   0

B.   1

C.   undefined

D.   2

E.   0.5

3. The function h(x) = $\dfrac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}$ is plotted below on the interval [-3 ,3]. The limit of h(x) as x approaches 0^+ is:

A.   0

B.   -1

C.   3

D.   1

E.   undefined

4. The function f(x) produced the following output

$$f(1.99) = 12.682 \qquad f(2.01) = 2.4975$$
$$f(1.999) = 12.968 \qquad f(2.001) = 2.4998$$
$$f(1.9999) = 12.997 \qquad f(2.0001) = 2.49998$$

Select **all** of the following statements that appear to be true:

A.   The function is continuous at x = 2.

B.   The limit of f(x) as x tends to 2 is 13.

C.   The limit of f(x) as x approaches 2^- is 2.5.

D.   The limit of f(x) as x tends to 2^+ is 2.5.